

Chiral Rings, Singularity Theory and Electric-Magnetic Duality

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We study in detail the space of perturbations of a pair of dual $N = 1$ supersymmetric theories based on an $SU(N_c)$ gauge theory with an adjoint X and fundamentals with a superpotential which is polynomial in X . The equivalence between them depends on non-trivial facts about polynomial equations, i.e. singularity theory. The classical chiral rings of the two theories are different. Quantum mechanically there are new relations in the chiral rings which ensure their equivalence. Duality interchanges “trivial” classical relations in one theory with quantum relations in the other and vice versa. We also speculate about the behavior of the theory without the superpotential.

1. Introduction and Summary.

The recent progress in the understanding of the dynamics of supersymmetric theories (for recent reviews and an extensive list of references see [1,2]) uncovered the crucial role played by electric-magnetic duality [3] in understanding the strong coupling dynamics. In scale invariant theories like the $N = 4$ [4] and finite $N = 2$ theories [5] duality provides two different descriptions of the same physical system that are equivalent at all distance scales. In asymptotically free theories the underlying degrees of freedom are visible at short distance and therefore the existence of different descriptions that are equivalent at all scales (exact duality) is impossible. Nevertheless, duality may be generalized to such theories [6], relating different quantum field theories with the same long distance behavior. When this long distance behavior is described by a non-trivial superconformal quantum field theory the dual theories are in the same universality class – they flow to the same fixed point of the renormalization group. When one of these theories is infra-red free it gives a simple description of the long distance physics of its strongly coupled dual.

No proof of this duality is known but there is a lot of evidence supporting it. There are three kinds of independent tests:

1. The two dual theories have the same global symmetries and the 't Hooft anomaly matching conditions for these symmetries are satisfied.
2. The two theories have the same moduli space of vacua. These are obtained by giving expectation values to the first components of chiral superfields.
3. The equivalence is preserved under deformations of the theories by the F -components of chiral operators. In particular the moduli spaces and chiral rings agree as a function of these deformations.

It is important to stress that in every one of these tests the classical theories are different and only the quantum theories become equivalent. There is also a crucial difference in the physical interpretation of the deformations of the two theories along the moduli space and by the chiral operators. Often, when one theory is Higgsed and becomes weaker, its dual is confining and becomes stronger. This is one of the reasons for interpreting the relation between these theories as electric-magnetic duality.

The last two tests above are closely related. The rings of chiral operators can be thought of as functions on the moduli space \mathcal{M}_0 . Hence, one might think that test two above implies test three. However, such a relation is not always simple. When there are points on the moduli space with extra massless particles the situation is more involved.

Then the moduli space \mathcal{M}_0 is constrained also by the equations of motion of these particles. As one adds sources to the theory proportional to F -components of chiral operators, the expectation values of the chiral fields can move away from \mathcal{M}_0 . The simplest way to describe the situation in such a case is to use an enlarged field space \mathcal{M} which includes all the fields (including those massive fields which become massless at special points) and to write a superpotential on \mathcal{M} . Then, the equations of motion derived from the superpotential lead to relations in the ring. These relations depend on the parameters in the theory – the sources.

In previously studied examples the structure of the chiral ring was relatively simple. In particular it was determined to a large extent by the symmetries. In general, the structure of the chiral ring can be quite involved. One may describe the ring in terms of generators satisfying certain relations. The relations in the classical chiral ring are consequences of the composite nature of the gauge invariant chiral operators. Quantum mechanically these relations can be modified [7]. We will see that it may also happen that new relations in the chiral ring appear quantum mechanically. Then, the classical chiral ring is truncated, in some cases rather dramatically.

The purpose of this paper is to study some qualitative and quantitative features of the duality of [6] in a class of examples that exhibit the phenomena mentioned above and a rich duality structure which helps in analyzing them. In the rest of this section we will describe the models we will study and state the main results. Derivations and many additional details appear in subsequent sections.

1.1. The models

The electric theory

Consider a $G = SU(N_c)$ gauge theory with an adjoint field X and N_f quarks Q^i and \tilde{Q}_i ($i, \tilde{i} = 1, \dots, N_f$) in the fundamental and anti-fundamental representations of the gauge group, respectively. These theories are still not fully understood (see however a discussion below in section 7). When a superpotential

$$W = \sum_{i=0}^k \frac{s_i}{k+1-i} \text{Tr } X^{k+1-i} \quad (1.1)$$

is turned on, the dynamics simplifies. At first sight the fact that the high order polynomials appearing in (1.1) can have any effect on the physics is surprising. Indeed, the presence of these non-renormalizable interactions seems irrelevant for the long distance behavior

of the theory, which will be our main interest below. Nevertheless, these operators have in general strong effects on the infrared dynamics. They are examples of operators that in the general theory of the renormalization group are known as *dangerously irrelevant*¹. Some comments about the properties of such operators appear in Appendix A.

To simplify the analysis of (1.1) with a traceless matrix X , we may view X as an arbitrary matrix and represent the constraint by a Lagrange multiplier term $\lambda \text{Tr } X$ in the superpotential. Physically, this amounts to adding two massive chiral fields to our problem, λ and $\text{Tr } X$. Clearly, this does not affect the long distance behavior. Then we can shift X by a term proportional to the identity matrix $X_s = X + \frac{s_1}{s_0 k} 1$ to set the coefficient of $\text{Tr } X_s^k$ in (1.1) to zero; λ is also shifted by a suitable constant. Such a shift removing the first subleading term in the superpotential is a standard manipulation in singularity theory. Rewriting the superpotential (1.1) in terms of the shifted X corresponds to performing an analytic reparametrization on the space of coupling constants. The new electric coupling constants will be denoted by $\{t_i\}$. The explicit coordinate transformation from $\{s_i\}$ to $\{t_i\}$ will appear below.

To find the classical moduli space of the theory one should first impose the D-flatness equations and mod out by gauge transformations. This is equivalent to modding out the space of chiral fields by $SU(N_c)\mathbb{C}$. Using this symmetry we can diagonalize X and then impose the equation of motion from (1.1). The eigenvalues of X must satisfy $W'(x) = 0$. For generic couplings $\{s_i\}$ there are k distinct solutions c_1, \dots, c_k . Vacua of the gauge theory are labeled by sequences of integers (r_1, r_2, \dots, r_k) ; r_l is the number of eigenvalues of the matrix $\langle X \rangle$ residing in the l 'th minimum of the potential. The gauge group is broken by the X expectation value:

$$SU(N_c) \rightarrow SU(r_1) \times SU(r_2) \times \dots \times SU(r_k) \times U(1)^{k-1} \quad (1.2)$$

At low energies the theory describes k decoupled supersymmetric QCD (SQCD) systems² with gauge groups $SU(r_l)$ and gauged baryon number. For a given choice of $\{r_l\}$ there is a moduli space of vacua associated with giving expectation values to the quarks. Therefore, the classical moduli space consists of many disconnected components parametrized by the

¹ We thank S. Shenker who pointed the relevance of this notion in this context.

² The different SQCD systems are in general coupled by high dimension operators which are sometimes important.

$\{r_i\}$. One can fine tune the couplings such that some of the eigenvalues $\{c_i\}$ coincide. The resulting multicritical behavior will be analyzed in section 2.

Quantum mechanically, not all vacua are stable. If, for example, we pick a classical vacuum (1.2) with one or more of the $r_l > N_f$, the resulting SQCD theory is destabilized by quantum effects [15,16]. Hence, such classical vacua are not present in the quantum moduli space. Similarly, some of the vacua in the multicritical case are destabilized and are removed from the quantum moduli space.

The classical chiral ring³ can be thought of as the ring generated by the operators $\text{Tr } X^j$ subject to two classes of constraints. The first comes from the equation of motion which follows from the superpotential (1.1), $W' = 0$. The second comes from the characteristic polynomial of X , and is an example of a relation following from the composite nature of gauge invariant operators mentioned above.

Quantum mechanically one expects on general grounds to find new relations in the chiral ring corresponding to the quantum reduction of the moduli space described above. It is in principle possible to construct these relations by requiring that imposing them has the effect of removing exactly the vacua that we know from our previous discussion should be removed, but this is very difficult in practice as well as unmotivated. Duality provides an elegant general solution to the problem, explaining why such new relations appear in the quantum chiral ring and providing a constructive way of determining them.

The magnetic theory.

It was shown in [8,9] that in the presence of a superpotential (1.1) there exists a simple dual magnetic description⁴. It is similar to the original electric theory but based on the gauge group $SU(\bar{N}_c = kN_f - N_c)$ with an adjoint field Y and N_f quarks q_i and \tilde{q}^i in the fundamental and anti-fundamental representations as well as some gauge invariant fields $(M_j)_i^i$ which correspond to the composite operators

$$(M_j)_i^i = \tilde{Q}_i X_s^{j-1} Q^i; \quad j = 1, \dots, k \quad (1.3)$$

where the suppressed color indices are summed over (when $s_1 \neq 0$ we find it convenient to define M_j in terms of X_s). The magnetic theory has a superpotential

$$W_{\text{mag}} = - \sum_l \frac{t_l}{k+1-l} \text{Tr } Y_s^{k+1-l} + \frac{1}{\mu^2} \sum_{l=0}^{k-1} t_l \sum_{j=1}^{k-l} M_j \tilde{q} Y_s^{k-j-l} q + \alpha_s(t) \quad (1.4)$$

³ Setting the quark fields to zero for the moment; the full structure will appear in section 2.

⁴ For related work see [10-14].

where the shifted field Y_s is defined similarly to X_s so that the coefficient of $\text{Tr } Y_s^k$ in the magnetic superpotential vanishes. The auxiliary scale μ is needed for dimensional reasons; note that even though the fields M_j are elementary in the magnetic description, the identification (1.3) implies that they are assigned scaling dimension $j+1$. Therefore, μ in (1.4) has indeed dimensions of mass. One could redefine M_j by powers of μ, s_0 to make their kinetic terms canonical. The numerical coefficients of the various terms in (1.4) can be calculated using flows, and will be derived below. The function $\alpha_s(t)$ will be computed too.

The magnetic superpotential (1.4) can be used to find the operator map relating the operators $\text{Tr } Y_s^j$ to the electric ones. Differentiating the generating functional with respect to the couplings t_{k+1-j} one can derive the map of the operators

$$\frac{1}{j} \text{Tr } X_s^j = \frac{\partial W}{\partial t_{k+1-j}} = \frac{\partial W_{\text{mag}}}{\partial t_{k+1-j}} . \quad (1.5)$$

The function α_s in (1.4) is important because it contributes to the map of the operators (1.5).

Note that (1.4) describes the duality map in the coordinates on theory space described above, $\{t_i\}$. Y_s is the adjoint field in those coordinates (see section 3). The relatively simple form of the map is in fact the main motivation behind the coordinate transformation $s \rightarrow t$. In the original s variables the duality map is significantly more complicated. Its precise form will be exhibited below.

The discussion of the classical and quantum moduli space and chiral ring for the theory (1.4) is essentially identical to the electric case, replacing $N_c, r_l \rightarrow \bar{N}_c, \bar{r}_l$. The classical electric and magnetic moduli spaces are different; they have different numbers of disconnected components (labeled by $\{r_i\}, \{\bar{r}_i\}$ respectively). Similarly, the classical chiral rings are different. Since the matrices X, Y are of different size, the effects of the characteristic polynomial are different in the two cases.

Quantum mechanically, the moduli spaces agree. After removing the subspaces of moduli space corresponding to unstable vacua we find, rather remarkably, the same number of components on both sides, with the same physical properties. Some of these properties will be investigated below.

Duality suggests a natural candidate for the quantum deformation of the chiral rings in the two theories. It is very natural to add to the classical relations coming from the equation of motion and the characteristic polynomial in the electric theory an additional

set of relations following from the *magnetic* characteristic polynomial using the duality map (1.5). These relations would be described as quantum strong coupling effects in the electric theory (we will see that they are trivial when the electric theory is weakly coupled), while in the magnetic language they correspond to classical ‘trivial’ relations following from compositeness of $\text{Tr } Y^j$. Similarly, the magnetic chiral ring is modified by quantum relations obtained via the duality map from the electric characteristic polynomial. One of our goals in this paper will be to establish that the quantum modification of the chiral ring just described does indeed take place.

The detailed map between the electric theory and the magnetic theory (1.4), (1.5) satisfies a number of non-trivial constraints:

1. The electric theory has a complicated vacuum structure which depends on the details of the superpotential (1.1). The magnetic theory should have the same vacuum structure.
2. The expectation values of the operators $\text{Tr } X^j$ can be calculated in all the vacua of the electric theory. They should agree with the corresponding calculation done using the magnetic variables and the map of operators (1.5).
3. As various perturbations are turned on, some fields become massive and can be integrated out both in the electric and in the magnetic theories. The two resulting low energy theories are calculable. They should be dual to one another. In particular, powerful constraints arise from requiring consistency of the operator map implied by duality with various deformations.

Additional constraints on duality follow from the requirement that certain scale matching relations are consistent with all deformations. These relations are discussed in the next subsection.

It is highly non-trivial and surprising that there exists a duality transformation which satisfies such a large number of consistency checks.

1.2. Scale matching.

An important chiral operator in the electric theory is the kinetic term of the gauge fields W_α^2 . Its coefficient is $P \log \Lambda$ where Λ is a dynamical scale related to the gauge coupling by dimensional transmutation, and P is the coefficient of the one loop beta function (in an $SU(N_c)$ gauge theory with N_f flavors of quarks in the fundamental representation $P = 3N_c - N_f$). Even though W_α is chiral (annihilated by $\bar{D}_{\dot{\alpha}}$), it is usually not a chiral

primary field at the IR fixed point. The reason is that the anomaly equation often⁵ relates it to another primary field \mathcal{O} as $W_\alpha^2 = \bar{D}^2 \mathcal{O}$. Therefore it is not in the chiral ring. Nevertheless, its coefficient in the electric Lagrangian should be related to its counterpart in the magnetic theory, $\bar{P} \log \bar{\Lambda}$ (\bar{P} is the coefficient of the one loop beta function in the magnetic theory). In our case we will show that the relation has the form

$$\Lambda^{2N_c - N_f} \bar{\Lambda}^{2\bar{N}_c - N_f} = \left(\frac{\mu}{s_0} \right)^{2N_f}. \quad (1.6)$$

The relation (1.6) shows in a quantitative way how one theory becomes stronger as the other becomes weaker. Its consistency with various flows will provide rather stringent quantitative tests of duality.

Λ ($\bar{\Lambda}$) is usually thought of as the scale at which the behavior of the electric (magnetic) theory crosses over from being dominated by the short distance fixed point to the long distance one (the typical mass scale of the theory). This raises a number of questions; it is not clear why Λ , $\bar{\Lambda}$ defined in such a way should satisfy a relation like (1.6) when the theories are only equivalent in the extreme infrared. Also, this definition leaves ambiguous a numerical factor, especially in theories with more than one scale.

One can think about eq. (1.6) and the scales appearing in it purely in the extreme IR theory. Since the scaling dimensions of various operators in the infrared are not the same as in the UV (see [6] and discussion below), one needs a dimensionful parameter to relate the UV operators to the IR ones. That dimensionful parameter, which can be defined for example through two point functions of such operators, is Λ . Similarly, in the magnetic theory one has $\bar{\Lambda}$, and additional parameters such as μ (1.4). The meaning of the scale matching relation is that $\bar{\Lambda}$ and μ must be chosen to obey (1.6) in order for the correlation functions of the two theories to agree including normalization. At any given point in the space of theories we can absorb the scales Λ , $\bar{\Lambda}$, μ into the definitions of the operators, thus making the scale matching relation (1.6) seem trivial. The non-trivial content in (1.6) is its consistency with duality under all possible deformations of both theories. Indeed, we will see that consistency leads to highly non-trivial checks of duality. The situation is reminiscent of the Zamolodchikov metric in two dimensional field theory, which is trivial at any given point in the space of theories, but whose curvature carries invariant geometrical information. The scale matching also describes an invariant relation between the geometries of the spaces of electric and magnetic theories.

⁵ This happens whenever the superpotential W vanishes, or more generally when W is independent of at least one of the matter superfields (which is the case in the class of theories considered here).

1.3. Outline

We plan to discuss two main issues. The first is the structure of the quantum chiral ring and moduli space, and their transformation under duality. The second is non-trivial quantitative tests of duality, which are possible because of the large number of vacua that the system possesses in general. In all these tests one uses symmetries to write down certain duality relations. This leaves in general some undetermined functions of the coupling constants. These functions can be calculated by assuming duality in some vacua of the theory. Since there are in general many more vacua (and independent tests) than unknown functions, the agreement of the resulting structures with duality in all vacua is a non-trivial check.

In section 2 we review the results of [8,9]. We describe the class of theories we will study, describe the duality map, and discuss their classical and quantum chiral rings and moduli spaces of vacua. In particular we establish the existence of new quantum relations in the chiral rings of these theories corresponding to classical relations in the duals.

In section 3 we construct the detailed map of the superpotential (1.4). We describe the transformation of coordinates $\{s\} \rightarrow \{t\}$ and show that the duality map is rather simple in the $\{t\}$ coordinates. After constructing this map we change coordinates back to the physically natural ones.

In section 4 we turn to the gauge coupling constant matching relation (1.6). We show that it is preserved under the various deformations. In particular, with arbitrary coupling constants t_i one can compute the effective Lagrangian of the low energy theory both in the electric and in the magnetic variables. These lead (generically) to dual pairs of supersymmetric QCD like theories whose scales are related by the appropriate scale matching relation. The couplings s_i can also be fine tuned to yield multicritical infrared behavior, whose consistency with duality puts further constraints on the structure described in sections 2, 3.

In section 5 we study the baryon operators in the theory and show that they are mapped correctly between the electric and magnetic theories. This provides additional non-trivial checks of duality and the explicit coefficients in the magnetic superpotential.

In section 6 we illustrate the general results with a few examples. In particular, we show that in some cases the quantum relations in the chiral ring lead to qualitative changes in the structure of the chiral ring. We conclude with some comments about the theory with no superpotential in section 7. Two appendices contain a discussion of dangerously irrelevant operators and some useful identities about polynomial equations which are used in the text.

2. Supersymmetric Yang-Mills theory coupled to adjoint and fundamental matter.

2.1. The electric theory.

We start with a review of the results of [8,9] on supersymmetric Yang-Mills theory with gauge group $G = SU(N_c)$ coupled to a single chiral matter superfield X_α^β in the adjoint representation of the gauge group, and N_f flavors of fundamental representation superfields, $Q_\alpha^i, \tilde{Q}_j^\beta$; $\alpha, \beta = 1, \dots, N_c$; $i, j = 1, \dots, N_f$. This theory is in a non-Abelian Coulomb phase for all $N_f \geq 1$. Its anomaly free global symmetry is

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_{R_1} \times U(1)_{R_2} \quad (2.1)$$

The two $SU(N_f)$ factors act by unitary transformations on Q, \tilde{Q} respectively; baryon number assigns charge $+1(-1)$ to $Q (\tilde{Q})$, while under the R symmetries the superspace coordinates θ_α are assigned charge 1, Q, \tilde{Q} charge B_f , and X charge B_a ; anomaly freedom implies that:

$$N_f B_f + N_c B_a = N_f. \quad (2.2)$$

Without a superpotential this model is not currently understood beyond the vicinity of $N_f \simeq 2N_c$ where perturbative techniques are reliable [17].

One of the main points of [8,9] was that the theory simplifies if we add a superpotential⁶

$$W = \frac{s_0}{k+1} \text{Tr } X^{k+1}. \quad (2.3)$$

This superpotential corresponds, for generic k , to a dangerously irrelevant perturbation of the theory with $W = 0$ (see Appendix A), and thus cannot be ignored despite being irrelevant near the (free) UV fixed point of the theory. This superpotential has the effect of truncating the chiral ring of the theory, imposing the constraint

$$(X^k)_\alpha^\beta - \frac{1}{N_c} (\text{Tr } X^k) \delta_\alpha^\beta = \text{D term} \quad (2.4)$$

which follows from the equation of motion for X ; it also removes many of the flat directions of the original theory. In addition, the superpotential (2.3) breaks one of the two R symmetries in (2.1). It is useful to think of s_0 (and other couplings to be introduced below) as background superfields whose lowest components get expectation values [18].

⁶ In [9] $s_0/(k+1)$ was denoted by g_k .

The superfield s_0 is then seen to transform under the $U(1)_R$ symmetries (2.2) with charge $B_0 = 2 - (k + 1)B_a$. Therefore, only the $U(1)_R$ symmetry under which the charge of the adjoint field X is $B_a = 2/(k + 1)$ leaves the vacuum with $\langle s_0 \rangle = s_0$ invariant, and corresponds to a good symmetry. The unbroken global symmetry of the model is:

$$SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \quad (2.5)$$

The matter fields Q , \tilde{Q} and X transform under this global symmetry as follows:

$$\begin{aligned} Q & \quad (N_f, 1, 1, 1 - \frac{2}{k+1} \frac{N_c}{N_f}) \\ \tilde{Q} & \quad (1, \overline{N}_f, -1, 1 - \frac{2}{k+1} \frac{N_c}{N_f}) \\ X & \quad (1, 1, 0, \frac{2}{k+1}). \end{aligned} \quad (2.6)$$

We will also be interested in more general superpotentials⁷:

$$W = \sum_{i=0}^{k-1} \frac{s_i}{k+1-i} \text{Tr} X^{k+1-i} + \lambda \text{Tr} X. \quad (2.7)$$

λ is a Lagrange multiplier enforcing the condition $\text{Tr} X = 0$. The coupling constant s_i has dimension $2 - k + i$ and $U(1)_R$ charge $B_i = 2 - (k + 1 - i)B_a$. For non zero $\{s_i\}$ (2.7) breaks both R symmetries (2.2). The space of theories labeled by the $\{s_i\}$ describes rather rich dynamics. Using transformations in $SU(N_c)\mathbb{C}$ one can diagonalize the matrix X . For generic $\{s_i\}$, such that

$$W'(x) = \sum_{i=0}^{k-1} s_i x^{k-i} + \lambda \equiv s_0 \prod_{i=1}^k (x - c_i) \quad (2.8)$$

with all eigenvalues c_i different from each other, the theory splits in the infrared into a set of decoupled SQCD theories. Ground states are labeled by sequences of integers $r_1 \leq r_2 \leq \dots \leq r_k$, where r_l is the number of eigenvalues of the matrix X residing in the l 'th minimum of the potential $V = |W'(x)|^2$. Clearly, $\sum_{l=1}^k r_l = N_c$. λ in (2.7) is determined by requiring that the sum of the eigenvalues (which depends on λ) vanishes,

$$\sum_{i=1}^k c_i r_i = 0. \quad (2.9)$$

⁷ It is standard in singularity theory to resolve a multicritical singularity such as (2.3) in order to study its properties.

In each vacuum X has a quadratic superpotential, i.e. it is massive and can be integrated out. The gauge group is broken by the X expectation value:

$$SU(N_c) \rightarrow SU(r_1) \times SU(r_2) \times \cdots \times SU(r_k) \times U(1)^{k-1} \quad (2.10)$$

$SU(r_l)$ is the gauge group of the l 'th decoupled SQCD theory one finds in the infrared⁸. The existence of a large number of vacua (2.10) (labeled by partitions of N_c into k or less integers) after resolving the singularity (2.7) is the key fact leading to simple but non-trivial quantitative checks of duality.

For generic $\{s_i\}$ (and fixed $\{r_i\}$) the classical infrared behavior of the theory, a decoupled set of SQCD theories with gauge groups (2.10), is insensitive to the precise values of $\{s_i\}$. Quantum mechanically, this is not the case; each of the low energy SQCD theories has a scale Λ_i , $i = 1, \dots, k$. These k scales are functions of the scale Λ of the underlying theory with the adjoint superfield, and the $k - 1$ couplings s_1, \dots, s_{k-1} . The functions $\Lambda_i(\Lambda, s_i)$ will be computed below (in section 4).

A similar discussion holds if the $\{s_l\}$ are fine tuned such that some of the c_i in (2.8) coincide:

$$W'(x) = \sum_{i=0}^{k-1} s_i \text{Tr} X^{k-i} + \lambda = s_0 \prod_{i=1}^m (x - c_i)^{n_i} \quad (2.11)$$

where $\sum_i n_i = k$; $m \leq k$; $n_i \geq 1$. Vacua are labeled by sequences (r_1, \dots, r_m) , $\sum_i r_i = N_c$, corresponding to different ways of partitioning the N_c eigenvalues among the m critical points c_i (2.11). In this case one finds in the infrared a decoupled set of theories with gauge groups $SU(r_l)$ and superpotentials of the form (2.3) with $k = n_l$ (see (2.11)). Therefore, the deformations (2.7) connect theories with different k in (2.3) and unify them into a single framework.

It is often possible to deduce non-trivial properties of the theory (2.3) by studying the deformed theories (2.7). As a simple example [9], turning on small s_i in (2.7) and using the fact that the resulting SQCD theories have stable vacua iff all $r_i \leq N_f$ one deduces that the theory (2.3) has a vacuum iff

$$N_f \geq \frac{N_c}{k}. \quad (2.12)$$

We will see other examples below.

⁸ Some of the r_l may vanish, in which case (2.10) is modified in an obvious way.

2.2. The magnetic theory

The main result of [8,9] was that the strongly coupled infrared physics of theory (2.3) can be studied using a dual description in terms of a “magnetic” supersymmetric gauge theory with gauge group $\bar{G} = SU(\bar{N}_c)$, $\bar{N}_c = kN_f - N_c$, and the following matter content: N_f flavors of (dual) quarks, $q_i^{\bar{\alpha}}$, $\tilde{q}_{\bar{\beta}}^j$, an adjoint superfield $Y_{\bar{\alpha}}^{\bar{\beta}}$, and gauge singlets $(M_j)_i^{\bar{i}}$ representing the generalized mesons,

$$(M_j)_i^{\bar{i}} = \tilde{Q}_i X^{j-1} Q^{\bar{i}}; \quad j = 1, 2, \dots, k \quad (2.13)$$

of the original, “electric” theory. The mesons M_j have in the magnetic theory standard kinetic terms $\int d^4\theta M_j^\dagger M_j$, rescaled by powers of s_0, μ . The magnetic superpotential is:

$$W_{\text{mag}} = \frac{\bar{s}_0}{k+1} \text{Tr } Y^{k+1} + \frac{s_0}{\mu^2} \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q. \quad (2.14)$$

The auxiliary scale μ (mentioned in the introduction) is needed for dimensional reasons. In the next sections we will see that it is very natural to normalize Y such that $\bar{s}_0 = -s_0$; this choice leads to all the coefficients⁹ in the sum over j in (2.14) being 1, as indicated.

The transformation properties of the magnetic matter fields, q, \tilde{q}, Y and M_j under the global symmetries (2.5) are:

$$\begin{aligned} q & \quad (\bar{N}_f, 1, \frac{N_c}{kN_f - N_c}, 1 - \frac{2}{k+1} \frac{kN_f - N_c}{N_f}) \\ \tilde{q} & \quad (1, N_f, -\frac{N_c}{kN_f - N_c}, 1 - \frac{2}{k+1} \frac{kN_f - N_c}{N_f}) \\ Y & \quad (1, 1, 0, \frac{2}{k+1}) \\ M_j & \quad (N_f, \bar{N}_f, 0, 2 - \frac{4}{k+1} \frac{N_c}{N_f} + \frac{2}{k+1}(j-1)) \end{aligned} \quad (2.15)$$

The auxiliary scale μ in (2.14) is actually not an independent parameter. The scale of the electric theory Λ , that of the magnetic theory $\bar{\Lambda}$, and μ satisfy the scale matching relation described in the introduction:

$$\Lambda^{2N_c - N_f} \bar{\Lambda}^{2\bar{N}_c - N_f} = C s_0^{-2N_f} \mu^{2N_f} \quad (2.16)$$

⁹ Which as we will see are uniquely determined by consistency of duality with deformations.

It is easy to check that (2.16) is invariant under all global symmetries including those under which s_0 , μ , Λ transform. It is in fact uniquely fixed by these symmetries. As in [19,2], the scale matching condition (2.16) implies that when the electric theory is weakly coupled the magnetic one is strongly coupled, and vice versa¹⁰. Differentiating the actions with respect to Λ holding s_0 and μ fixed, the electric gauge field strength W_α^2 is related to the magnetic one \bar{W}_α^2 by the relation:

$$\bar{W}_\alpha \bar{W}^\alpha = -W_\alpha W^\alpha. \quad (2.17)$$

It is highly non-trivial, and will be shown in section 4, that the scale matching relation (2.16) which is completely determined by global symmetries, is consistent with all possible deformations of the theory, such as turning on masses for the quarks Q^i , giving expectation values to the mesons M_j (2.13), and deforming the theory to non-zero s_i (2.7). The numerical constant C in (2.16) is also fixed by the flows, $C = 1$, and its value and in particular its independence of N_f , N_c , k lead to additional tests of duality.

We view the consistency of (2.16) with deformations as strong evidence for the validity of the electric-magnetic duality hypothesis of [6,8,9].

When the electric superpotential is deformed to (2.7) the electric theory develops a large number of vacua labeled by (r_1, \dots, r_m) , with $\sum r_i = N_c$ (see the discussion after (2.11)). The $SU(r_l)$ theory contains an adjoint field with superpotential $W \simeq X^{n_l+1}$. The magnetic theory has a similar structure obtained by analyzing vacua of the superpotential (1.4). Its vacua are labeled by integers $(\bar{r}_1, \dots, \bar{r}_m)$, $\sum \bar{r}_i = \bar{N}_c$. Classically, the moduli spaces do not agree. However [8,9] when we include the quantum stability constraints (2.12) we find a one to one correspondence of the quantum vacua in the two theories. The vacuum map is non trivial:

$$(r_1, r_2, \dots, r_m) \leftrightarrow (n_1 N_f - r_1, n_2 N_f - r_2, \dots, n_m N_f - r_m). \quad (2.18)$$

A class of operators that was discussed in [8,9], and will be revisited in section 5, are the baryon-like operators:

$$B^{i_1 \dots i_{n_1}, j_1 \dots j_{n_2}, \dots, z_1 \dots z_{n_k}} = \epsilon^{\alpha_1 \dots \alpha_{n_1}, \beta_1 \dots \beta_{n_2}, \dots, \rho_1 \dots \rho_{n_k}} Q_{\alpha_1}^{i_1} \dots Q_{\alpha_{n_1}}^{i_{n_1}} (XQ)_{\beta_1}^{j_1} \dots (XQ)_{\beta_{n_2}}^{j_{n_2}} \dots (X^{k-1}Q)_{\rho_1}^{z_1} \dots (X^{k-1}Q)_{\rho_{n_k}}^{z_{n_k}} \quad (2.19)$$

¹⁰ As an example, the electric theory is weakly coupled when N_f is slightly below $2N_c$, and free for $N_f \geq 2N_c$. The magnetic theory is strongly coupled in that whole region.

where $\sum_i n_i = N_c$. The duality map relating the baryons B (2.19), and the analogously defined dual baryons \bar{B} is:

$$B^{i_1 \dots i_{n_1}, j_1 \dots j_{n_2}, \dots, z_1 \dots z_{n_k}} = P \left(\prod_{i=1}^k \frac{1}{\bar{n}_i!} \right) \bar{s}_0^{\frac{k N_f}{2}} (\mu^2)^{-\frac{\bar{N}_c}{2}} \Lambda^{\frac{k}{2}(2N_c - N_f)} \quad (2.20)$$

$$\epsilon^{i_1 \dots i_{n_1}, \bar{z}_1 \dots \bar{z}_{\bar{n}_k}} \epsilon^{j_1 \dots j_{n_2}, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1}}} \dots \epsilon^{z_1 \dots z_{n_k}, \bar{i}_1 \dots \bar{i}_{\bar{n}_1}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_1}, \dots, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_k}}$$

where $\bar{n}_l = N_f - n_{k+1-l}$, $l = 1, \dots, k$, and P is a phase that will be discussed in section 5. The form of (2.20) is determined by global symmetries, while the overall numerical constant is uniquely fixed by the flows, and it is non-trivial that it is consistent with the deformations, such as (2.7). This compatibility will be established in section 5.

In addition to the mesons (2.13) and baryons (2.19), the chiral ring of the electric (magnetic) theory contains the generators $\text{Tr} X^j$ ($\text{Tr} Y^j$), $j = 2, \dots, k$. Global symmetries require (up to terms depending on the quarks which we will determine below) that

$$\text{Tr} Y^j = f_j \text{Tr} X^j \quad (2.21)$$

with f_j calculable numerical constants. In the next section we will calculate f_j and describe the generalization of (2.21) to the deformed theories (2.7).

The duality map described in this section may be used to study the physics of the theory (2.3) at strong coupling using a weak coupling description in terms of magnetic variables. For example, one finds that this theory may exhibit at strong coupling a “free magnetic phase,” with the magnetic variables governed by a non-asymptotically free gauge theory. Some additional features are described in [8,9].

2.3. Classical and quantum chiral rings.

The classical chiral ring of the electric theory is generated by the generalized mesons M_j (2.13), baryons (2.19) and traces of the adjoint matrix, $\text{Tr} X^j$. The latter, which we will focus on here satisfy classically two sets of constraints, following from the equation of motion $W' = 0$, (2.8) and from the characteristic polynomial. Defining $f(p) \equiv \det(p - X)$, the constraint is $f(X) = 0$. The two together give rise to a set of algebraic equations for the generators of the classical chiral ring. The solutions of these equations describe the ring of functions on the classical moduli space.

One can repeat the same construction for the magnetic theory. The discussion proceeds in complete parallel to that of the moduli space. Classically the chiral rings of the

electric and magnetic theories cannot be the same: while the equation of motion $W' = 0$ gives similar relations for the generators, the characteristic polynomials give different sets of constraints, since the sizes of the matrices X and Y are different. However, the discussion of the correspondence of the quantum moduli spaces points to the correct modification of the chiral rings in the quantum theory needed to restore duality and the relation between the (quantum) chiral rings and moduli spaces. For generic values of the couplings $\{s_i\}$ the quantum restriction on the $\{r_i\}$, $r_l \leq n_l N_f$ turns under duality (2.18) to the trivial condition $\bar{r}_l \geq 0$ (and vice versa). This means that if we add the *classical* characteristic polynomial relations on the magnetic chiral ring as *quantum* relations in the electric theory, we are guaranteed that they will have precisely the right effect on the electric moduli space, eliminating the unstable electric vacua and leaving all other parts of the electric moduli space unchanged.

To summarize, we have the following structure of the electric and magnetic chiral rings. Both are generated by the operators $\text{Tr } X^j$ (or $\text{Tr } Y^j$ – the two sets of operators are related by the transformation (1.5) which we will make more explicit below) satisfying three sets of constraints:

1. The equation of motion, $W' = 0$.
2. The vanishing of the electric characteristic polynomial, $f_{\text{el}} = 0$.
3. The vanishing of the magnetic characteristic polynomial, $f_{\text{mag}} = 0$.

The first set of relations appears classically in both the electric and magnetic theories and gives similar constraints in both. The second set of relations is classical in the electric theory (and is due to the compositeness of $\text{Tr } X^j$), but it is a non-trivial quantum effect in the magnetic theory. The third is classical in the magnetic theory and quantum in the electric one.

3. The mapping of the superpotential.

To complete the discussion of the previous section we must construct the duality map taking the electric theory (2.3) to the magnetic one (2.14) when the electric theory is deformed to (2.7). One expects the magnetic superpotential (2.14) to be deformed as well, and in this section we will discuss the detailed way in which this happens. We will start with a discussion of the deformation of the first term on the r.h.s. of (2.14), the superpotential for Y , and then turn to the second term, proportional to M_j . In the process we will verify all the numerical coefficients in (1.4) and learn some qualitative things about the duality map. We start with a discussion of the Y superpotential.

3.1. The problem.

The electric superpotential (2.7) describes a space of theories parametrized by the couplings s_i . On general grounds one expects a magnetic superpotential

$$\bar{W} = \sum_{i=0}^{k-1} \frac{\bar{s}_i}{k+1-i} \text{Tr} Y^{k+1-i} + \bar{\lambda} \text{Tr} Y + \alpha(s). \quad (3.1)$$

$\bar{s}_i = \bar{s}_i(s)$ are the magnetic coupling constants and $\alpha(s)$ is a constant. The purpose of this section is to find $\bar{s}(s)$ and $\alpha(s)$.

In complete analogy with the discussion of the electric theory in section 2, the magnetic theory (3.1) exhibits for generic \bar{s}_i a large number of vacua, parametrized by integers \bar{r}_l corresponding to the number of eigenvalues of the matrix Y with the value \bar{c}_l , the l 'th minimum of the magnetic superpotential $|\bar{W}'|^2$. The $\{\bar{c}_l\}$ are defined by a magnetic analogue of (2.8). Clearly $\sum_l \bar{r}_l = \bar{N}_c = kN_f - N_c$. The low energy magnetic theory is a direct product of decoupled copies of SQCD with N_f flavors of quarks, with the gauge group broken according to:

$$SU(\bar{N}_c) \rightarrow SU(\bar{r}_1) \times SU(\bar{r}_2) \times \cdots \times SU(\bar{r}_k) \times U(1)^{k-1} \quad (3.2)$$

The picture proposed in [9] was that the original duality between (2.3) and (2.14) reduces for the deformed theories (2.7), (3.1) to a direct product of the SQCD dualities of [6] for the separate factors in (2.10), (3.2). That means that the magnetic multiplicities $(\bar{r}_1, \dots, \bar{r}_k)$ are related to the electric ones via the SQCD duality relation (compare to (2.18)):

$$\bar{r}_i = N_f - r_i. \quad (3.3)$$

The fact that (3.3) is a one to one map of the sets of vacua of the electric and magnetic theories follows from results of [15,16] on vacuum stability in SQCD.

Furthermore, it was argued in [9] that when two or more of the critical points of W coincide, as in (2.11), the same number of critical points of \bar{W} should coincide. If, using the notation of equation (2.11), the order of a critical point c_i (and therefore that of \bar{c}_i as well) is n_i , the degeneracies r_i and \bar{r}_i of this critical point in the electric and magnetic theories are related by

$$\bar{r}_i = n_i N_f - r_i. \quad (3.4)$$

The duality of section 2 induces in this case a duality of a similar kind, between an electric theory with gauge group $SU(r_i)$ and a superpotential $\text{Tr } X^{n_i+1}$, and a magnetic one with gauge group $SU(\bar{r}_i) = SU(n_i N_f - r_i)$ with a similar superpotential.

For the above scenario to be realized, the electric and magnetic superpotentials must be closely related. In particular, the fact that whenever any number of critical points c_i coincide, the same number of dual critical points \bar{c}_i must coincide as well is a very strong constraint on the dual couplings $\bar{s}_i(s)$. A naive guess for the solution would be a proportionality relation between the electric and magnetic couplings,

$$\bar{s}_i = c s_i \quad (3.5)$$

with c a constant. Throughout this paper we will be using the convention

$$\bar{s}_0 = -s_0 \quad (3.6)$$

which defines the normalization of Y relative to X . This convention would set $c = -1$. Eq. (3.5) implies that $\bar{c}_i = c_i$ and automatically satisfies the degeneration of singularities requirement described above. However, the mapping $\bar{s}_i(s)$ cannot (in general) be as simple as (3.5) because of the non-trivial mapping of electric to magnetic multiplicities, (3.3). Indeed, in a vacuum with given $\{r_l\}$, the tracelessness of X implies that $\sum_{l=1}^k c_l r_l = 0$, whereas, assuming $\bar{c}_i = c_i$ and using (3.3), the tracelessness of Y is the condition $\sum_{l=1}^k c_l (N_f - r_l) = 0$. The two are incompatible unless s_1 in (2.7) vanishes. For non zero s_1 we conclude that the mapping $\bar{s}(s)$ must be non trivial; we will construct it below.

The duality map relating $\text{Tr } X^j$ to $\text{Tr } Y^l$ is closely related to the mapping $\bar{s}(s)$. Define the free energy of the model as

$$e^{-\int d^4x d^2\theta F(s_i) + c.c.} = \langle e^{-\int d^4x d^2\theta W(X, s_i) + c.c.} \rangle \quad (3.7)$$

where s_i are background chiral superfields. Then, correlation functions of the operators $\text{Tr } X^j$ are given by derivatives of the free energy F with respect to the superfields s_i :

$$\frac{1}{k+1-i} \langle \text{Tr } X^{k+1-i} \rangle = \frac{\partial F}{\partial s_i} \quad (3.8)$$

and similarly in the magnetic theory, in terms of the dual free energy \bar{F} :

$$\frac{1}{k+1-i} \langle \text{Tr } Y^{k+1-i} \rangle = \frac{\partial \bar{F}}{\partial \bar{s}_i} \quad (3.9)$$

where duality implies

$$\bar{F}(\bar{s}_i(s)) = F(s_i). \quad (3.10)$$

Since $\text{Tr } X^j$, $\text{Tr } Y^j$ are tangent vectors to the space of theories, (3.8), (3.9) we find that the electric and magnetic operators are related by:

$$\frac{1}{k+1-i} \text{Tr } X^{k+1-i} = \sum_j \frac{\partial \bar{s}_j}{\partial s_i} \frac{1}{k+1-j} \text{Tr } Y^{k+1-j} + \frac{\partial \alpha}{\partial s_i} \quad (3.11)$$

Taking the expectation values of both sides of eq. (3.11) we find that the mapping $\bar{s}_i(s)$ must satisfy a constraint in addition to the previously described one on the degeneration of eigenvalues. The expectation values of the left and right hand sides of (3.11) which depend in a highly non-trivial way on the particular vacuum chosen (the set of r_i) must satisfy a relation that is *independent* of the particular vacuum chosen. Clearly, the special form of the mapping (3.11) and the large number of vacua in which it should hold presents a formidable constraint on their form.

3.2. A reparametrization of the space of theories and a general solution.

It is convenient to think of X, Y as general $U(N)$ matrices, with a dynamical Lagrange multiplier λ ($\bar{\lambda}$) imposing the tracelessness of X (Y). Consider the electric theory, described by (2.7). It is convenient to define a shifted X , denoted by X_s as:

$$X_s \equiv X + b\mathbf{1} \quad (3.12)$$

with

$$b = \frac{s_1}{s_0 k}. \quad (3.13)$$

The shift (3.12), (3.13) cancels the first subleading term in W , leading to the superpotential:

$$W_s(X_s) = \sum_{i=0}^{k-1} \frac{t_i}{k+1-i} \text{Tr } X_s^{k+1-i} + \lambda_s (\text{Tr } X_s - bN_c) + \beta N_c \quad (3.14)$$

where $W_s(X_s) \equiv W(X)$,

$$\begin{aligned} t_i &= \sum_{j=0}^i \binom{k-j}{i-j} (-b)^{i-j} s_j \\ \lambda_s &= \lambda + \sum_{j=0}^{k-1} (-b)^{k-j} s_j \\ \beta &= - \sum_{j=0}^{k-1} \frac{k-j}{k+1-j} (-b)^{k+1-j} s_j. \end{aligned} \quad (3.15)$$

Note that $t_0 = s_0$ and $t_1 = 0$. The transformation (3.12), (3.13) corresponds to an analytic coordinate transformation on the space of theories (2.7). A similar transformation can be performed on (3.1), replacing Y by Y_s and $\bar{\lambda}$ with $\bar{\lambda}_s$. The $k-1$ independent couplings s_i , $i = 1, \dots, k-1$ are replaced by the $k-2$ couplings t_i , $i = 2, \dots, k-1$, and b (3.13). In the X_s variables the coefficient of the first subleading term $\text{Tr } X_s^k$ in the superpotential (3.14) always vanishes. The information about that coefficient in the original description (2.7) is in b (3.13). In a sense, the transformation (3.12), (3.13) allowed us to trade the operator $\text{Tr } X^k$ for the operator λ , which is possible since by the X equation of motion (ignoring D terms as usual)

$$\lambda = \frac{1}{N_c} \sum_{i=0}^{k-2} s_i \text{Tr } X^{k-i}. \quad (3.16)$$

Since in the t_i , b parametrization of the space of theories the first subleading term in W vanishes by construction, it is natural to postulate that the duality map for the eigenvalues of X_s , a_i defined analogously to (2.8) is simply

$$\bar{a}_i = a_i; \quad i = 1, \dots, k. \quad (3.17)$$

With the convention (3.6) this means that (compare to (3.5)):

$$\begin{aligned} \bar{t}_i &= -t_i \\ \bar{\lambda}_s &= -\lambda_s. \end{aligned} \quad (3.18)$$

The second equation in (3.18) is an operator identity; it can be thought of as arising from the coupling relations:

$$\bar{b}\bar{N}_c = -bN_c; \quad \alpha_s = \text{independent of } b \quad (3.19)$$

using (3.11). Equations (3.18), (3.19) specify the mapping $\bar{s}_i(s)$ completely.

We now determine α_s . Using (3.11), (3.18) we find the operator equation

$$\text{Tr } X_s^{k+1-i} = -\text{Tr } Y_s^{k+1-i} + (k+1-i) \frac{\partial \alpha_s}{\partial t_i} \quad (3.20)$$

($i = 2, \dots, k-1$). The expectation values of the l.h.s of (3.20) in a vacuum specified by a set of $\{r_l\}$ is

$$\text{Tr } X_s^{k+1-i} = \sum_{l=1}^k r_l a_l^{k+1-i} \quad (3.21)$$

while in the magnetic theory:

$$\text{Tr } Y_s^{k+1-i} = \sum_{l=1}^k \bar{r}_l \bar{a}_l^{k+1-i} = \sum_{l=1}^k (N_f - r_l) a_l^{k+1-i} = -\text{Tr } X_s^{k+1-i} + N_f u_{k+1-i} \quad (3.22)$$

where (see also Appendix B):

$$u_j \equiv \sum_{i=1}^k a_i^j. \quad (3.23)$$

Comparing (3.20) and (3.22) we see that for consistency of the picture advocated above we must be able to write the u_j as:

$$u_{k+1-j} = \frac{k+1-j}{N_f} \frac{\partial \alpha_s}{\partial t_j} \quad (3.24)$$

Indeed, one can check that eq. (3.24) is satisfied with α_s given by:

$$\alpha_s = \frac{N_f}{k+1} \sum_{i=2}^{k-1} i t_i \frac{u_{k+1-i}}{k+1-i}. \quad (3.25)$$

The proof of the fact that (3.25) satisfies (3.24) uses the following property of the u_n (3.23):

$$\frac{\partial}{\partial t_j} \frac{u_{j+l}}{j+l} = \text{independent of } j. \quad (3.26)$$

This and other properties of the u_n are reviewed in Appendix B.

Summarizing, the main results of this subsection are the mapping of the couplings in the electric superpotential t_i , b defined by (3.15), (3.13) to their magnetic counterparts, given by equations (3.18), (3.19). This simple transformation law induces a transformation (3.20) for the operators $\text{Tr } X_s^j$ with $j = 2, \dots, k-1$. The operator $\text{Tr } X^k$ that was conjugate to the coupling s_1 is eliminated in favor of the Lagrange multiplier λ which also has a simple transformation given by the second equation in (3.18).

Of course, the simple transformation laws described in this subsection become more complicated when we translate them back to the original coordinates s_i .

3.3. The duality map in the original variables.

After the discussion of the previous subsection it is not difficult to describe the duality map for the perturbed superpotential (2.7) in the original coordinates s_i . The main point is that while as we shall see the mapping $\bar{s}_i(s)$ (and therefore the operator map (3.11)) is

somewhat complicated in this case, the mapping of the eigenvalues remains simple. Indeed, using the simple relation between the eigenvalues of X_s and of Y_s (3.17) and the relation between X and Y and their shifted counterparts X_s and Y_s (3.12), (3.13) we conclude that the mapping of the eigenvalues of X , Y , (c_i, \bar{c}_i) is:

$$\bar{c}_i = c_i + d; \quad d \equiv \frac{s_1 N_f}{s_0 \bar{N}_c}. \quad (3.27)$$

Similarly, we derive the map of the coupling constants

$$\bar{s}_m = - \sum_{l=0}^m s_{m-l} (-d)^l \binom{k-m+l}{l} \quad (3.28)$$

and the operators (using (3.11)):

$$\text{Tr } Y^j = - \sum_{i=2}^j \binom{j}{i} d^{j-i} \text{Tr } X^i + N_f \bar{u}_j - N_c d^j; \quad j < k \quad (3.29)$$

$$\begin{aligned} \text{Tr } Y^k = & - \sum_{i=2}^k \binom{k}{i} d^{k-i} \text{Tr } X^i + \frac{k N_f}{N_c} \sum_{j=0}^{k-2} \frac{s_j}{s_0} \text{Tr } X^{k-j} \\ & + \bar{N}_c d^k + N_f \sum_{j=1}^{k-1} \binom{k}{j} d^{k-j} u_j - N_f \sum_{l=1}^{k-1} \frac{s_l}{s_0} u_{k-l}. \end{aligned} \quad (3.30)$$

The function α defined in (3.11) can be expressed in terms of β and α_s defined in (3.15), (3.25) as:

$$\alpha(s_i) = \beta N_c - \bar{\beta} \bar{N}_c + \alpha_s(t_i(s)). \quad (3.31)$$

Interestingly, when all s_i except s_0 vanish (i.e. the superpotential is (2.3)) there is nevertheless a non-trivial operator matching following from equations (3.29), (3.30):

$$\begin{aligned} \text{Tr } Y^j &= - \text{Tr } X^j; \quad j = 2, \dots, k-1 \\ \text{Tr } Y^k &= \frac{\bar{N}_c}{N_c} \text{Tr } X^k \end{aligned} \quad (3.32)$$

From eq. (3.32) one can read off the values of the coefficients c_j of section 2 (2.21).

3.4. The M_j terms in the magnetic superpotential.

So far our discussion focused on the way the first term in the magnetic superpotential (2.14) is deformed as we deform the electric superpotential (2.7). In this subsection we will use these results to determine the deformation of the second term in W_{mag} . We will work in the parametrization of the space of theories described in subsection 3.2.

When one turns on non-vanishing couplings t_i in (2.7), the magnetic superpotential (2.14) can in principle receive contributions proportional to t_j , $t_j t_l$, etc, consistently with the global symmetries. The way to fix all these terms is to require that duality act in the way described after eq. (3.2). Namely, for generic t_i we expect the magnetic theory to split into an approximately *decoupled* set of SQCD theories that are dual to the different decoupled factors in (2.10).

This requirement of decoupling is rather non-trivial since the second term on the r.h.s. of (2.14) tends to couple the different $SU(\bar{r}_i)$ theories. Indeed, denote the first r_1 components (in color) of the electric quarks¹¹ Q by Q_1 , the next r_2 by Q_2 and so on. Similarly, the first \bar{r}_1 components of q are denoted by q_1 , the next \bar{r}_2 by q_2 , etc.. Then expanding around $\langle X_s \rangle$ we find

$$M_j = \tilde{Q}_1 Q_1 a_1^{j-1} + \tilde{Q}_2 Q_2 a_2^{j-1} + \cdots + \tilde{Q}_k Q_k a_k^{j-1} \quad (3.33)$$

(recall that for generic s_i we defined $M_j = \tilde{Q} X_s^{j-1} Q$) $\tilde{Q}_l Q_l$ are the mesons of the l 'th electric SQCD theory with gauge group $SU(r_l)$. The color $SU(r_l)$ indices are as usual suppressed and summed over. Similarly we write:

$$\tilde{q} Y_s^j q = \tilde{q}_1 q_1 a_1^j + \tilde{q}_2 q_2 a_2^j + \cdots + \tilde{q}_k q_k a_k^j \quad (3.34)$$

where as in 3.2 we denote the shifted Y field appropriate for the \bar{t}_i coordinate system on theory space by Y_s . In the above formula we used the fact that in the coordinates t_i the duality map is trivial, $\bar{a}_i = a_i$ (3.17).

The second term in W_{mag} (2.14) has to be corrected in such a way that the different SQCD theories do not couple – there should not be any cross terms coupling $\tilde{q}_j q_j$, $\tilde{Q}_i Q_i$ with $i \neq j$. The unique solution to this requirement is

$$W_{\text{mag}} = \sum_l \frac{\bar{t}_l}{k+1-l} \text{Tr } Y_s^{k+1-l} + \frac{1}{\mu^2} \sum_{l=0}^{k-1} t_l \sum_{j=1}^{k-l} M_j \tilde{q} Y_s^{k-j-l} q. \quad (3.35)$$

¹¹ In this subsection flavor indices will be suppressed.

All the numerical coefficients in the second term on the r.h.s. of (3.35) are fixed by the requirement that when we substitute (3.33), (3.34) into it, cross terms such as $\tilde{Q}_1 Q_1 \tilde{q}_2 q_2$ vanish. Indeed, the coefficient of the above operator is proportional to:

$$\sum_{l=0}^{k-1} t_l \sum_{j=1}^{k-l} a_1^{j-1} a_2^{k-j-l} = \sum_{l=0}^{k-1} t_l \frac{a_2^{k-l} - a_1^{k-l}}{a_2 - a_1} = 0 \quad (3.36)$$

which vanishes because a_1, a_2 are roots of W' (see (2.8)). Thus, with the choice of couplings in (3.35) the magnetic theory reduces for generic t_i into *decoupled* SQCD theories as required by duality¹².

The form (3.35) which at this stage of the discussion is completely fixed, must satisfy additional consistency conditions. The simplest of these involves getting the right behavior when some of the roots a_i coincide. For example, if $a_1 = a_2$ and all other a_i are different, (3.33) is replaced by:

$$M_j = a_1^{j-1} \tilde{Q}_1 Q_1 + (j-1) a_1^{j-2} \tilde{Q}_1 X_{s1} Q_1 + a_3^{j-1} \tilde{Q}_3 Q_3 + \cdots + a_k^{j-1} \tilde{Q}_k Q_k. \quad (3.37)$$

with X_{s1} the fluctuating deviation of the shifted adjoint field of $SU(r_1)$ from its v.e.v. Similarly, (3.34) is replaced by:

$$\tilde{q} Y_s^j q = a_1^j \tilde{q}_1 q_1 + j a_1^{j-1} \tilde{q}_1 Y_{s1} q_1 + a_3^j \tilde{q}_3 q_3 + \cdots + a_k^j \tilde{q}_k q_k. \quad (3.38)$$

Using (3.37), (3.38) in (3.35) we find the correct superpotential for decoupled SQCD and an $SU(\bar{r}_1)$ sector with $k = 2$ as required by the duality. All these checks give the expected results. More generally, when some eigenvalues a_i coincide as in (2.11), one finds that the terms that must vanish are always proportional to derivatives of W at a_i which vanish.

Additional consistency conditions on the detailed form of (3.35) will appear in the next section. We see again that consistency of the deformed theory with duality fixes uniquely coefficients in the superpotential of the unperturbed theory (2.14).

It is also useful to note at this point that the t_l dependence of the magnetic superpotential (3.35) implies a correction to the dual of $\text{Tr } X_s^j$ given in (3.20). Indeed, differentiating the free energies of the electric and magnetic theories (see (1.5), (3.7) – (3.11)) one finds:

$$\text{Tr } X_s^{k+1-i} = -\text{Tr } Y_s^{k+1-i} + \frac{k+1-i}{\mu^2} \sum_{j=1}^{k-i} M_j \tilde{q} Y_s^{k-j-i} q + (k+1-i) \frac{\partial \alpha_s}{\partial t_i} \quad (3.39)$$

($i = 2, \dots, k-1$). The second term on the r.h.s. mixes the operators $\text{Tr } X_s^j$ with the generalized magnetic mesons.

¹² When Y is integrated out there can be more terms of higher dimension in the low energy superpotential which we do not discuss.

4. Consistency of scale matching with deformations.

In section 2 we mentioned the relation (2.16) between the scale of the electric theory, Λ , that of the magnetic theory, $\bar{\Lambda}$, and the dimensionful parameter μ entering (2.14), (3.35):

$$\Lambda^{2N_c - N_f} \bar{\Lambda}^{2\bar{N}_c - N_f} = s_0^{-2N_f} \mu^{2N_f} \quad (4.1)$$

It is interesting to check whether this relation is consistent with the various deformations that the model possesses. These include adding terms proportional to M_j (2.13) to the superpotential, and turning on s_i (2.7). In this section we will check the compatibility of (4.1) with two kinds of flows:

1. Adding a mass term to one of the flavors (e.g. $m(M_1)_{N_f}^{N_f}$).
2. The general s_i perturbations.

In doing that we should stress that the relation (4.1) does not depend on the masses or s_i . This follows from the symmetries.

We will see that, remarkably, there is complete detailed agreement of the two kinds of flows with (4.1). We start with a summary of the conventions we will be using.

4.1. Conventions.

Before discussing the flows we must specify the threshold corrections relating the scales of the theory when massive particles are integrated out. When we integrate out a massive fundamental chiral superfield Q with mass m , the $SU(N_c)$ gauge theory with an adjoint and N_f flavors goes in the infrared to one with an adjoint and $N_f - 1$ flavors, and:

$$\Lambda_{N_c, N_f}^{2N_c - N_f} m = \Lambda_{N_c, N_f - 1}^{2N_c - (N_f - 1)} \quad (4.2)$$

The analogous relation in SQCD is:

$$\Lambda_{N_c, N_f}^{3N_c - N_f} m = \Lambda_{N_c, N_f - 1}^{3N_c - (N_f - 1)} \quad (4.3)$$

When we integrate out a chiral adjoint field X of mass m , we have:

$$m^{N_c} \Lambda_{N_c, N_f}^{2N_c - N_f} = \Lambda_{N_c, N_f}^{3N_c - N_f} \quad (4.4)$$

where the scale on the l.h.s. corresponds to the theory with an adjoint, and the scale on the r.h.s to SQCD. Finally, when we integrate out a massive vector superfield of mass m

in the fundamental representation of the gauge group (i.e. when part of the gauge group is Higgsed) we have in the theory with an adjoint and N_f fundamentals:

$$\Lambda_{N_c, N_f}^{2N_c - N_f} = m^2 \Lambda_{N_c - 1, N_f}^{2N_c - N_f - 2} \quad (4.5)$$

whereas in SQCD:

$$\Lambda_{N_c, N_f}^{3N_c - N_f} = m^2 \Lambda_{N_c - 1, N_f - 1}^{3N_c - N_f - 2} \quad (4.6)$$

Notice that in the theory with the adjoint (4.5) N_f does not decrease under Higgsing. This is because one massless chiral superfield is eaten by the massive gauge field but another appears from decomposing the adjoint of $SU(N_c)$ w.r.t. $SU(N_c - 1)$.

Out of (4.2) – (4.6) only three definitions are independent (e.g. (4.2), (4.4), (4.5)). With these conventions the scale matching condition in SQCD is [2]:

$$\Lambda_{\text{SQCD}}^{3N_c - N_f} \bar{\Lambda}_{\text{SQCD}}^{3\bar{N}_c - N_f} = (-)^{N_f - N_c} \mu^{N_f} \quad (4.7)$$

Here μ is an auxiliary scale similar to that in (2.14). It is defined such that the magnetic superpotential in SQCD is :

$$W_{\text{mag}}^{\text{SQCD}} = \frac{1}{\mu} M \tilde{q} q. \quad (4.8)$$

With the conventions in hand we next turn to examine the flows.

4.2. The mass flow.

Consider adding to the electric theory a mass term:

$$W_{\text{el}} = \frac{s_0}{k+1} \text{Tr } X^{k+1} + m \tilde{Q}_{N_f} Q^{N_f}. \quad (4.9)$$

The theory loses a flavor in the infrared; the scales of the high and low energy theories (Λ_{N_c, N_f} and $\Lambda_{N_c, N_f - 1}$ respectively) are related by (4.2):

$$\Lambda_{N_c, N_f}^{2N_c - N_f} = \frac{1}{m} \Lambda_{N_c, N_f - 1}^{2N_c - (N_f - 1)}. \quad (4.10)$$

In the magnetic theory, the superpotential (2.14) is modified to:

$$W_{\text{mag}} = \frac{\bar{s}_0}{k+1} \text{Tr } Y^{k+1} + \frac{s_0}{\mu^2} \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q + m (M_1)_{N_f}^{N_f}. \quad (4.11)$$

One next needs to set the massive fields to solutions of their equations of motion and integrate them out. It is easy to see that all the fields $(M_j)_i^{N_f}$, $(M_j)_i^{N_f}$ ($i = 1, \dots, N_f$),

and components of q , \tilde{q} , Y in k of the \bar{N}_c directions are massive. The expectation values of \tilde{q}^{N_f} , q_{N_f} , Y satisfy:

$$\tilde{q}^{N_f} Y^{l-1} q_{N_f} = -\delta_{l,k} \frac{m\mu^2}{s_0}; \quad l = 1, \dots, k \quad (4.12)$$

Taking into account the D terms (which fix the relative normalization of \tilde{q} , q , Y) and Y equation of motion leads to:

$$\begin{aligned} \tilde{q}_\alpha^{N_f} &= \delta_{\alpha,1} \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{1}{k+1}}; \\ q_{N_f}^\alpha &= \delta^{\alpha,k} \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{1}{k+1}}; \\ Y_\beta^\alpha &= \begin{cases} \delta_{\beta-1}^\alpha \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{1}{k+1}} & \beta = 1, \dots, k \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (4.13)$$

We can think of the effect of m in (4.11) in two stages. First, the magnetic gauge group $SU(\bar{N}_c)$ is broken by the Higgs mechanism to $SU(\bar{N}_c - k)$. At this stage, q_{N_f} , \tilde{q}^{N_f} , Y_m^α $m = 2, \dots, k$, Y_α^s $s = 1, \dots, k-1$ ($\alpha = k+1, \dots, \bar{N}_c$) gain a mass and join k massive vector superfields in the fundamental representation of the unbroken, $SU(\bar{N}_c - k)$ gauge group. According to our convention (4.5) this generates a factor of the k 'th power of the mass squared of the vector superfields, $\left[\left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{2}{k+1}} \right]^k$.

In a second stage, Y_1^α and Y_α^k get a mass from expanding the superpotential:

$$W_{\text{mag}} = \frac{\bar{s}_0}{k+1} \text{Tr } Y^{k+1} \simeq \bar{s}_0 \langle Y \rangle^{k-1} Y_1^\alpha Y_\alpha^k = \bar{s}_0 \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{k-1}{k+1}} Y_1^\alpha Y_\alpha^k. \quad (4.14)$$

In (4.14) we have used the fact that in expanding $\text{Tr } Y^{k+1}$ to leading order in $Y_1^\alpha Y_\alpha^k$ we must take these two Y 's next to each other; terms like $\text{Tr } \langle Y^n \rangle Y_1 \langle Y^m \rangle Y^k$ ($n, m \neq 0$) do not contribute such mass terms. Since Y_1^α , Y_α^k can be thought of as massive chiral superfields in the fundamental representation of $SU(\bar{N}_c - k)$ with mass $\bar{s}_0 \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{k-1}{k+1}}$ (see (4.14), (4.13)), we use (4.2) for the scale matching.

Finally, combining the two stages we have:

$$\bar{\Lambda}_{\bar{N}_c, N_f}^{2\bar{N}_c - N_f} = \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{2k}{k+1}} \frac{1}{\bar{s}_0 \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{k-1}{k+1}}} \bar{\Lambda}_{\bar{N}_c - k, N_f - 1}^{2(\bar{N}_c - k) - N_f} = \frac{m\mu^2}{\bar{s}_0^2} \bar{\Lambda}_{\bar{N}_c - k, N_f - 1}^{2(\bar{N}_c - k) - N_f} \quad (4.15)$$

Using (4.10), (4.15) we conclude that (4.1) leads to:

$$\frac{1}{m} \Lambda_{N_c, N_f-1}^{2N_c-(N_f-1)} \frac{m\mu^2}{\bar{s}_0^2} \bar{\Lambda}_{\bar{N}_c-k, N_f-1}^{2(\bar{N}_c-k)-N_f} = \mu^{2N_f} s_0^{-2N_f} \quad (4.16)$$

or:

$$\Lambda_{N_c, N_f-1}^{2N_c-(N_f-1)} \bar{\Lambda}_{\bar{N}_c-k, N_f-1}^{2(\bar{N}_c-k)-N_f} = \mu^{2(N_f-1)} s_0^{-2(N_f-1)} \quad (4.17)$$

which is exactly the right scale matching relation for the theory with $N_f - 1$ flavors. We therefore conclude that the scale matching relation (4.1) is consistent with the mass perturbation (4.9).

An interesting element of the preceding analysis is the fact that the overall relative coefficient between the first and second terms in the magnetic superpotential (2.14) was important for quantitative agreement of the matching conditions. This relative coefficient was not fixed by the discussion in section 3.4 and we can view the analysis of this subsection as a way to determine it. We will soon see a non-trivial check of its value from the s_i flows (2.7).

4.3. Deformation of the X superpotential: the generic case.

In this subsection we will be deforming the superpotential for X (Y) in the electric (magnetic) theory, as in (2.7), (3.1). Recall the duality

$$\begin{aligned} W(X_s) &= \sum_{i=0}^{k-1} \frac{t_i}{k+1-i} \text{Tr } X_s^{k+1-i} \\ \bar{W}(Y_s) &= \sum_{i=0}^{k-1} \frac{\bar{t}_i}{k+1-i} \text{Tr } Y_s^{k+1-i} + \alpha_s \\ \bar{t}_i &= -t_i; \quad \bar{a}_i = a_i \end{aligned} \quad (4.18)$$

where a_i are the eigenvalues of X_s defined as in (2.8) and \bar{a}_i are similarly related to Y_s .

To test (4.1) in the deformed theory (4.18) we proceed in two stages. First consider generic t_i such that all eigenvalues a_i (2.8) are distinct. Then the IR electric theory is a direct product of decoupled SQCD theories with gauge groups $SU(r_i)$ (see discussion following (2.8)). The scale of the i 'th SQCD theory is related to that of the high energy theory by:

$$\Lambda^{2N_c-N_f} = \Lambda_i^{3r_i-N_f} \left[\prod_{j \neq i} (a_i - a_j)^{2r_j} \right] \frac{1}{[W''(a_i)]^{r_i}} \quad (4.19)$$

One can think of (4.19) as following from a two step process. First we turn on an expectation value for X_s :

$$\langle X_s \rangle = \text{diag}(a_1^{r_1}, a_2^{r_2}, \dots, a_k^{r_k}) \quad (4.20)$$

where $a_1^{r_1}$ means the eigenvalue a_1 appears r_1 times, etc. This makes some vector fields massive and using (4.5) we get the factor of $\left[\prod_{j \neq i} (a_i - a_j)^{2r_j}\right]$ in (4.19). Then, at a second stage, we integrate out the massive adjoint field in the $SU(r_i)$ theory using (4.4). This gives rise to $\frac{1}{[W''(a_i)]^{r_i}}$.

We evaluate $W''(a_i)$ using (2.8) and find for Λ_i :

$$\Lambda_i^{3r_i - N_f} = \Lambda^{2N_c - N_f} t_0^{r_i} \prod_{j \neq i} (a_i - a_j)^{r_i - 2r_j} \quad (4.21)$$

Repeating the same arguments for the magnetic theory using (4.18) we find:

$$\bar{\Lambda}_i^{3\bar{r}_i - N_f} = \bar{\Lambda}^{2\bar{N}_c - N_f} t_0^{\bar{r}_i} \prod_{j \neq i} (a_i - a_j)^{\bar{r}_i - 2\bar{r}_j}. \quad (4.22)$$

At this stage we have no independent check on (4.21), (4.22) separately (although one will appear in the next section when we discuss baryons), but multiplying the left and right hand sides of (4.21) and (4.22) and using (4.1), (4.7) we find

$$\Lambda_i^{3r_i - N_f} \bar{\Lambda}_i^{3\bar{r}_i - N_f} = (-)^{N_f - r_i} \mu_i^{N_f} = (-)^{N_f - r_i} \frac{\mu^{2N_f}}{t_0^{N_f}} \prod_{j \neq i} (a_i - a_j)^{-N_f} \quad (4.23)$$

which means that

$$\mu_i = \frac{\mu^2}{t_0} \frac{1}{\prod_{j \neq i} (a_i - a_j)}. \quad (4.24)$$

Recall that μ_i is defined through the magnetic superpotential in the $SU(\bar{r}_i)$ theory (compare to (4.8)):

$$W_{\text{mag}}^{(i)} = \frac{1}{\mu_i} \tilde{Q}_i Q_i \tilde{q}_i q_i. \quad (4.25)$$

The scale parameters μ_i can be independently calculated by the analysis described in section 3.4. By decomposing M_j , $\tilde{q} Y^l q$ into their $SU(r_i)$, $SU(\bar{r}_i)$ components (3.33), (3.34) and evaluating the coefficient of $\tilde{Q}_i Q_i \tilde{q}_i q_i$ in the magnetic superpotential (3.35) we find:

$$\frac{1}{\mu_i} = \frac{1}{\mu^2} \sum_{l=0}^{k-1} t_l \sum_{j=1}^{k-l} a_1^{j-1} a_1^{k-j-l} = \sum_{l=0}^{k-1} (k-l) t_l a_1^{k-l-1} = \frac{1}{\mu^2} W''(a_i) = \frac{t_0}{\mu^2} \prod_{j \neq i} (a_i - a_j) \quad (4.26)$$

which is exactly the right value (4.24). Since the magnetic superpotential (3.35) is completely fixed by the considerations of section 3, the agreement between (4.24) and (4.26) gives another non-trivial quantitative check of duality.

We conclude that, at least for generic t_i , the deformation (4.18) is consistent with the scale matching relation (4.1). This still leaves the question of whether we get a consistent picture when some $\{a_i\}$ coincide, which we briefly address in the next subsection.

4.4. Deformation of the X superpotential: coinciding eigenvalues.

Consider for simplicity the special case discussed in section 3.4, where two of the a_i , say a_1 and a_2 , coincide¹³. Then instead of k copies of SQCD, one gets in the IR $k - 2$ copies of SQCD, corresponding to a_3, \dots, a_k , and one copy of the model (2.3) with $k = 2$. The discussion of the $k - 2$ SQCD vacua is exactly as in the last subsection. The only new feature here is the discussion of the scale of the $SU(r_1)$ $k = 2$ model corresponding to a_1 , and its dual, the $SU(\bar{r}_1)$ model with $k = 2$ and $\bar{r}_1 = 2N_f - r_1$. The scale Λ_1 of that low energy model is related to Λ by:

$$\Lambda_1^{2r_1 - N_f} = \Lambda^{2N_c - N_f} \prod_{j=3}^k (a_j - a_1)^{-2r_j} \quad (4.27)$$

by an argument analogous to that following (4.19). Similarly:

$$\bar{\Lambda}_1^{2\bar{r}_1 - N_f} = \bar{\Lambda}^{2\bar{N}_c - N_f} \prod_{j=3}^k (a_j - a_1)^{-2\bar{r}_j} \quad (4.28)$$

Defining $t_0^{(l)}$ and $\mu^{(l)}$ to be the analogues of t_0 , μ in the low energy theory, we have the scale matching relation:

$$\Lambda_1^{2r_1 - N_f} \bar{\Lambda}_1^{2\bar{r}_1 - N_f} = \left(t_0^{(l)}\right)^{-2N_f} \left(\mu^{(l)}\right)^{2N_f} \quad (4.29)$$

Multiplying (4.27) and (4.28) and using (4.1), (4.29) we find:

$$\frac{t_0^{(l)}}{\mu^{(l)}} = \frac{t_0}{\mu} \prod_{j=3}^k (a_j - a_1) \quad (4.30)$$

¹³ The discussion can be trivially extended to the general case.

One can again perform an independent check of (4.30) by calculating $t_0^{(l)}$, $\mu^{(l)}$ directly. Starting with $t_0^{(l)}$ we use (2.8) near $x \simeq a_1$:

$$W'(x) = (x - a_1)^2 \prod_{j=3}^k (x - a_j) \simeq t_0 \left[\prod_{j=3}^k (a_1 - a_j) \right] (x - a_1)^2 \equiv t_0^{(l)} (x - a_1)^2 \quad (4.31)$$

so that:

$$t_0^{(l)} = t_0 \prod_{j=3}^k (a_1 - a_j). \quad (4.32)$$

To get $\mu^{(l)}$ we use the decompositions (3.37), (3.38) and calculate the coefficient of $\tilde{Q}_1 Q_1 \tilde{q}_1 Y_{s1} q_1$ (or equivalently $\tilde{Q}_1 X_{s1} Q_1 \tilde{q}_1 q_1$). We find that

$$\frac{t_0}{\mu^2} \prod_{i=1}^k (a_1 - a_i) = \frac{t_0^{(l)}}{(\mu^{(l)})^2}; \text{ or } : \mu^{(l)} = \mu. \quad (4.33)$$

Combining (4.32), (4.33) we see that (4.30) is indeed valid and the structure of flows (2.7) is consistent with the scale matching relation (4.1).

5. The mapping of baryons in the deformed theory.

In section 2 we proposed an exact mapping (2.20) between the baryon operators in the electric and magnetic theories. The mapping involves powers of the dimensionful couplings that are fixed by global symmetries and numerical factors that can be fixed by consistency with the various flows. In this section we will outline the checks of consistency of (2.20) with various deformations. We will see that these consistency checks lead to additional highly non-trivial checks of duality. In particular, since Λ appears in the map (2.20), we will have a more direct check on the expressions for the scales Λ_i (4.21), (4.22) derived in section 4.

There is an inherent phase ambiguity in (2.20) related to the possibility of making field redefinitions in the two theories corresponding to global symmetries of the problem. Performing a baryon number transformation in the magnetic theory introduces an arbitrary phase in (2.20) and an opposite phase in the relation between \tilde{B} and its dual $\tilde{\tilde{B}}$. We can use this freedom to make the phases in the mappings of B and \tilde{B} identical. This still leaves a sign ambiguity in all our formulae for mapping of B , \tilde{B} separately, which disappears in the mapping of $B\tilde{B}$. We will not write explicitly these \pm signs, but instead leave the

branches of square roots appearing in B and \tilde{B} unspecified. Phase factors appearing in the matching of $B\tilde{B}$ have an absolute meaning and therefore will be calculated.

In the first subsection we will discuss the consistency of (2.20) when turning on masses for quarks (4.9). In the second subsection we will verify the consistency of (2.20) with deformations of the superpotential $W(X)$ (2.7).

5.1. The mass flow.

As in section 4 we add a mass term for Q^{N_f}, \tilde{Q}_{N_f} (4.9). The electric theory now has $N_f - 1$ flavors. The electric baryon (2.19) splits into components with the indices $i_{p_1}, j_{p_2}, \dots, z_{p_k}$ taking values between 1 and $N_f - 1$ which stay massless, and the rest that become massive. In the magnetic theory we deform the superpotential by an mM_1 term (4.11). The gauge group breaks from $SU(\bar{N}_c)$ to $SU(\bar{N}_c - k)$ and the number of flavors decreases by one, $N_f \rightarrow N_f - 1$. One notes that:

1. It follows from (4.13) that:

$$\langle (Y^j q^{N_f})^\alpha \rangle = \delta_{k-j}^\alpha \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{j+1}{k+1}} \quad (5.1)$$

2. All components of the magnetic quarks q, \tilde{q} and the adjoint field Y with color indices between 1 and k , as well as q^{N_f}, \tilde{q}_{N_f} (all color components) become massive.
3. Due to the structure of the r.h.s. of (2.20) if any one or more of the indices i_{p_1}, \dots, z_{p_k} equals N_f the corresponding magnetic baryon (the r.h.s. of (2.20)) is massive, since it is not possible to saturate the color indices $\bar{\alpha} = 1, \dots, k$ corresponding to broken generators except with massive quarks. This is in agreement with the behavior of the electric theory (the l.h.s. of (2.20)) as mentioned above.
4. If all indices i_{p_1}, \dots, z_{p_k} are less than N_f the r.h.s. of (2.20) does lead to massless baryons in the magnetic theory. The broken color indices $\bar{\alpha} = 1, \dots, k$ can now be saturated by the expectation values of the $Y^j q^{N_f}$. The only way to saturate the broken color indices by the expectation values (5.1) is to replace in \bar{B} (2.20) one of each group of indices \bar{i}, \dots, \bar{z} by the appropriate expectation value in (5.1). This leads to the following transformation of the r.h.s. of (2.20) under the mass flow:

$$\begin{aligned} \epsilon^{i_1 \dots i_{n_1}, \bar{z}_1 \dots \bar{z}_{\bar{n}_k}} \epsilon^{j_1 \dots j_{n_2}, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1}}} \dots \epsilon^{z_1 \dots z_{n_k}, \bar{i}_1 \dots \bar{i}_{\bar{n}_1}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_1}, \dots, \bar{y}_1 \dots \bar{y}_{\bar{n}_{k-1}}, \bar{z}_1 \dots \bar{z}_{\bar{n}_k}}^{\bar{N}_c, N_f} &\rightarrow (-)^{\frac{k(k-1)}{2}} \\ \bar{n}_1 \bar{n}_2 \dots \bar{n}_k \left(\frac{m\mu^2}{\bar{s}_0} \right)^{\frac{k}{2}} \epsilon^{i_1 \dots i_{n_1}, \bar{z}_1 \dots \bar{z}_{\bar{n}_k-1}} \dots \epsilon^{z_1 \dots z_{n_k}, \bar{i}_1 \dots \bar{i}_{\bar{n}_1-1}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{n}_1-1}, \dots, \bar{z}_1 \dots \bar{z}_{\bar{n}_k-1}}^{\bar{N}_c-k, N_f-1} & \end{aligned} \quad (5.2)$$

The factors of \bar{n}_i come from the number of possibilities of placing the expectation value among the $\bar{n}_i X^{i-1}Q$ operators. The phase is due to a certain reordering of the k color indices $1, \dots, k$ that is needed to bring the baryon into standard form after symmetry breaking. A similar factor does not appear in the analogous mapping for \tilde{B} . Clearly, the only meaningful quantity is the phase appearing in the transformation of $B\tilde{B}$.

Inserting (5.2) into (2.20) we see that all the factors generated in the breaking magically arrange to give the form (2.20) again, with $N_f \rightarrow N_f - 1$. In particular, one notes that $\bar{n}_i \rightarrow \bar{n}_i - 1$, and the scale absorbs the factor of m (see (4.2)). Eq. (5.2) and the analogous relation following from the analysis of the mapping (2.20) under turning on an expectation value for M_k in the electric theory allow one to determine the dependence on N_f and N_c of the unambiguously defined phase in the mapping relation for $B\tilde{B}$. One finds that the phase, called P in (2.20) is:

$$P^2 = (-)^{\frac{k(k-1)}{2}N_f - N_c}. \quad (5.3)$$

As discussed above, only P^2 , which is a sign, is meaningful.

5.2. The baryon mapping in the presence of a deformed superpotential.

When a general superpotential (2.7) is present the gauge group breaks generically to (2.10). Correspondingly, the baryon operator (2.19) can be expressed in terms of the baryons of the SQCD theories (2.10). Similarly, the baryon of the magnetic $SU(\bar{N}_c)$ theory is decomposable into the baryons of the magnetic SQCD theories (3.2). The mapping of baryons in the high energy theory, (2.20), should reduce in the deformed theory to the baryon mapping in the individual SQCD theories:

$$B^{i_1 \dots i_{N_c}} = \frac{1}{\bar{N}_c!} \Lambda^{\frac{1}{2}(3N_c - N_f)} \bar{\mu}^{\frac{1}{2}(N_f - N_c)} \epsilon^{i_1 \dots i_{N_c} \bar{i}_1 \dots \bar{i}_{N_f - N_c}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{\bar{N}_c}} \quad (5.4)$$

Eq. (5.4) has the same sign ambiguity discussed above.

The general discussion of the reduction of (2.20) under the deformations (2.7) is unfortunately rather complicated. To get a flavor of the issues involved we will discuss here the special case of a theory with a cubic superpotential ($k = 2$ in (2.7)); furthermore, we will only discuss the special baryons with $n_1 = N_f$, $n_2 = N_c - N_f$ (see (2.19)). The main reason for considering these baryons is that they are dual by (2.20) to the simplest magnetic baryons, those with $\bar{n}_1 = 2N_f - N_c = \bar{N}_c$ and $\bar{n}_2 = 0$. Using (2.20) and the

phase P found in the previous subsection we expect for these baryons the following duality mapping:

$$B^{j_1 \dots j_{n_2}} \equiv \frac{1}{N_f!} \epsilon_{i_1 \dots i_{N_f}} \epsilon^{\alpha_1 \dots \alpha_{N_f}, \beta_1 \dots \beta_{N_c - N_f}} Q_{\alpha_1}^{i_1} \dots Q_{\alpha_{N_f}}^{i_{N_f}} (XQ)_{\beta_1}^{j_1} \dots (XQ)_{\beta_{n_2}}^{j_{n_2}} = \frac{P}{\bar{N}_c!} \bar{s}_0^{N_f} \mu^{-\bar{N}_c} \Lambda^{2N_c - N_f} \epsilon^{j_1 \dots j_{n_2}, \bar{i}_1 \dots \bar{i}_{N_f - n_2}} \epsilon_{\bar{\alpha}_1 \dots \bar{\alpha}_{\bar{N}_c}} q_{\bar{i}_1}^{\bar{\alpha}_1} \dots q_{\bar{i}_{\bar{N}_c}}^{\bar{\alpha}_{\bar{N}_c}} \quad (5.5)$$

We will check the validity of (5.5) when a general superpotential $W(X) = \frac{s_0}{3} \text{Tr } X^3 + \frac{s_1}{2} \text{Tr } X^2$ is introduced. In this particular case W has two critical points a_1, a_2 , and the electric and magnetic color groups are broken as in (2.10), (3.2) to: $SU(r_1) \times SU(r_2) \times U(1)$ and $SU(\bar{r}_1) \times SU(\bar{r}_2) \times U(1)$ respectively. The consistency check of (5.5) involves comparing two paths:

1. Decompose $B^{j_1 \dots j_{n_2}}$ in terms of the electric baryons of SQCD and then use the SQCD mapping (5.4) to obtain an expression in terms of the magnetic SQCD baryons.
2. Use (5.5) first, and then decompose the magnetic baryon into the SQCD baryons of the broken theory.

Clearly, the two expressions obtained this way should coincide for agreement with duality. The electric baryon decomposes after the breaking as:

$$B^{j_1 \dots j_{n_2}} = \frac{1}{N_f!} \sum_{s=0}^{r_1} (-)^{(N_f-s)(r_1-s)} a_1^{r-s} a_2^{N_f - N_c - (r_1-s)} \binom{N_f}{s} \binom{N_f - N_c}{r_1 - s} \epsilon_{i_1 \dots i_{N_f}} \epsilon^{\alpha_1 \dots \alpha_{r_1}} Q_{1, \alpha_1}^{i_1} \dots Q_{1, \alpha_s}^{i_s} Q_{1, \alpha_{s+1}}^{j_1} \dots Q_{1, \alpha_{r_1}}^{j_{r_1-s}} \epsilon^{\beta_1 \dots \beta_{r_2}} Q_{2, \beta_1}^{i_{s+1}} \dots Q_{2, \beta_{N_f-s}}^{i_{N_f}} Q_{2, \beta_{N_f-s+1}}^{j_{r_1-s+1}} \dots Q_{2, \beta_{r_2}}^{j_{n_2}} \quad (5.6)$$

Q_1, Q_2 here are the SQCD quarks defined in section 3.5. The non-trivial phases are due to the rearrangement of the ϵ symbol of $SU(N_c)$ in the order corresponding to a product of ϵ symbols of $SU(r_1) \times SU(r_2)$. Next we replace the electric SQCD baryons in (5.6) by their magnetic counterparts, using (5.4):

$$\epsilon^{\alpha_1 \dots \alpha_{r_1}} Q_{1, \alpha_1}^{i_1} \dots Q_{1, \alpha_s}^{i_s} Q_{1, \alpha_{s+1}}^{j_1} \dots Q_{1, \alpha_{r_1}}^{j_{r_1-s}} = \epsilon^{i_1 \dots i_s j_1 \dots j_{r_1-s} \bar{i}_1 \dots \bar{i}_{N_f - r_1}} \bar{B}_{\bar{i}_1 \dots \bar{i}_{N_f - r_1}}^{(1)} \frac{1}{(N_f - r_1)!} \Lambda_1^{\frac{1}{2}(3r_1 - N_f)} \bar{\mu}_1^{\frac{1}{2}(N_f - r_1)} \quad (5.7)$$

and an analogous relation for the baryon constructed out of Q_2 . The scales Λ_i , and the parameters μ_i are given by (4.21), (4.22). Another s dependent phase (in addition to that in (5.6)) appears from the rearrangement of the flavor indices in the various ϵ

tensors. After summing over the i flavor indices the sum over s can be performed producing $(a_1 - a_2)^{N_f - N_c}$. Putting all the factors together we obtain:

$$B^{j_1 \cdots j_{n_2}} = (-)^{N_f - N_c} \frac{N_f!}{(N_f - r_1)!(N_f - r_2)!} \bar{s}_0^{N_f} \mu^{-\bar{N}_c} \Lambda^{2N_c - N_f} \epsilon^{j_1 \cdots j_{n_2}, \bar{j}_1 \cdots \bar{j}_{N_f - r_1}, \bar{l}_1 \cdots \bar{l}_{N_f - r_2}} \bar{B}_{\bar{j}_1 \cdots \bar{j}_{N_f - r_1}}^{(1)} \bar{B}_{\bar{l}_1 \cdots \bar{l}_{N_f - r_2}}^{(2)} \quad (5.8)$$

It is straightforward to see that the same result for $B^{j_1 \cdots j_{n_2}}$ is obtained by decomposing the r.h.s. of eq. (5.5) into baryons of $SU(\bar{r}_1)$ and $SU(\bar{r}_2)$.

6. Examples.

In this section we will study some of the consequences of the general phenomena discussed in previous sections in some particular cases.

6.1. The case $k = 2$.

We start with the general cubic superpotential, (2.7) with $k = 2$. The only coupling that exists in this case is s_1 . For generic s_1 ,

$$W = \frac{s_0}{3} \text{Tr } X^3 + \frac{s_1}{2} \text{Tr } X^2 \quad (6.1)$$

W has two critical points, c_1, c_2 . Vacua are labeled by integers r_1, r_2 corresponding to placing r_1 of the eigenvalues in the first critical point, and the remaining $r_2 = N_c - r_1$ in the second one. For $r_1 \neq 0, N_c$ the gauge group breaks to (compare to (2.10)):

$$SU(N_c) \rightarrow SU(r_1) \times SU(r_2) \times U(1). \quad (6.2)$$

Eq. (2.8) and the tracelessness condition can be replaced by the two linear equations

$$\begin{aligned} c_1 + c_2 &= -\frac{s_1}{s_0} \\ r_1 c_1 + r_2 c_2 &= 0 \end{aligned} \quad (6.3)$$

with solution

$$\begin{aligned} c_1 &= \frac{s_1}{s_0} \frac{r_2}{r_1 - r_2} \\ c_2 &= \frac{s_1}{s_0} \frac{r_1}{r_2 - r_1} \end{aligned} \quad (6.4)$$

The chiral ring is generated by $\text{Tr } X^2$, whose expectation value in a vacuum with multiplicities (r_1, r_2) is:

$$\langle \text{Tr } X^2 \rangle = r_1 c_1^2 + r_2 c_2^2 = N_c \left(\frac{s_1}{2s_0} \right)^2 \left[\frac{N_c^2}{(r_1 - r_2)^2} - 1 \right] \quad (6.5)$$

Similarly, in the magnetic theory:

$$\begin{aligned} \bar{c}_1 &= \frac{\bar{s}_1}{\bar{s}_0} \frac{\bar{r}_2}{\bar{r}_1 - \bar{r}_2} \\ \bar{c}_2 &= \frac{\bar{s}_1}{\bar{s}_0} \frac{\bar{r}_1}{\bar{r}_2 - \bar{r}_1} \end{aligned} \quad (6.6)$$

and

$$\langle \text{Tr } Y^2 \rangle = \bar{r}_1 \bar{c}_1^2 + \bar{r}_2 \bar{c}_2^2 = \bar{N}_c \left(\frac{\bar{s}_1}{2\bar{s}_0} \right)^2 \left[\frac{\bar{N}_c^2}{(\bar{r}_1 - \bar{r}_2)^2} - 1 \right] \quad (6.7)$$

Comparing to (3.11) we see that to match the r dependent terms in $\langle \text{Tr } X^2 \rangle$ and $\langle \text{Tr } Y^2 \rangle$ we should choose

$$\bar{s}_1 = s_1 \frac{N_c}{\bar{N}_c} \quad (6.8)$$

which is a special case of (3.19). Comparing the constant terms of (6.5), (6.7) then leads to

$$\alpha(s) = \frac{N_c}{24} \frac{s_1^3}{s_0^2} \left(\frac{N_c^2}{\bar{N}_c^2} - 1 \right) \quad (6.9)$$

Eq. (6.9) is a special case of (3.31). The operator relation (3.11) takes in this case the form:

$$\text{Tr } X^2 = \frac{N_c}{\bar{N}_c} \text{Tr } Y^2 + \frac{N_c}{4} \frac{s_1^2}{s_0^2} \left(\frac{N_c^2}{\bar{N}_c^2} - 1 \right). \quad (6.10)$$

which should be compared to (3.30). There are two things to note here:

1. Considering the deformed theory with $s_1 \neq 0$ allows one to uniquely determine the numerical coefficients in the operator mapping (3.11) including the numerical constants f_j (2.21) which are defined in the theory with $s_1 = 0$.
2. While c_i (6.4) and \bar{c}_i (6.6) depend on r_i , they satisfy a simple linear relation

$$\bar{c}_i = c_i + \frac{s_1 N_f}{s_0 \bar{N}_c}. \quad (6.11)$$

This linear relation is a special case of (3.27).

For even N_c we encounter here an example of an interesting general phenomenon which occurs whenever N_c and k are not relatively prime. The solution for the eigenvalues c_1 and

c_2 (6.4) is singular when $r_1 = r_2 = N_c/2$ and $s_1 \neq 0$ and therefore these vacua are absent. On the other hand, for $s_1 = 0$ the values of c_1 and c_2 seem to be ambiguous. Indeed, in this case the superpotential has a flat direction with $\langle X \rangle = \text{diag}(c, \dots, c, -c, \dots, -c)$ (up to gauge transformations). The physics along this flat direction is exactly that of the “missing vacuum” with $r_1 = r_2 = N_c/2$. The adjoint field is massive and the gauge symmetry is broken: $SU(N_c) \rightarrow [SU(N_c/2)]^2 \times U(1)$. The only difference is that the mass of the adjoint field and the scales of the two SQCD theories depend on the Higgs parameter c instead of s_1 .

How does the chiral ring look in this case? The only chiral operator in the electric theory that can be written in terms of X alone using the equation of motion (2.4) is $\text{Tr } X^2$. The electric characteristic polynomial gives rise to a relation for X that depends on the parity of N_c . For odd N_c this relation takes the form:

$$(\text{Tr } X^2)^{\frac{N_c+1}{2}} + b_1(\text{Tr } X^2)^{\frac{N_c-1}{2}} + \dots + b_{\frac{N_c-1}{2}}(\text{Tr } X^2) = 0 \quad (6.12)$$

where b_l are easily calculable coefficients which depend on s_1/s_0 . When $s_1 = 0$ all b_l go to zero, and the relation is $(\text{Tr } X^2)^{\frac{N_c+1}{2}} = 0$ in agreement with the fact that moduli space is in this case a point (setting $M_j = 0$ as usual). For generic s_1 (6.12) has $(N_c + 1)/2$ solutions in one to one correspondence with the moduli space (6.5).

For even N_c the relation following from the characteristic polynomial is

$$s_1 \left((\text{Tr } X^2)^{\frac{N_c}{2}} + b_1(\text{Tr } X^2)^{\frac{N_c-2}{2}} + \dots + b_{\frac{N_c}{2}}(\text{Tr } X^2) \right) = 0. \quad (6.13)$$

For non zero s_1 there are $N_c/2$ solutions in one to one correspondence with the solutions of (6.5). In particular, the solution with $r_1 = r_2$ which is singular does not appear. For $s_1 = 0$ the characteristic polynomial (6.13) does not provide any constraints on $\text{Tr } X^2$, in agreement with the presence of the flat direction described above.

Quantum mechanically, we have learned that one needs to add to (6.12) (or (6.13)) another relation which is obtained from the characteristic polynomial of the magnetic theory using the operator map (6.10). For odd N_c , $\bar{N}_c = 2N_f - N_c$ is odd too, and one finds a relation

$$(\text{Tr } Y^2)^{\frac{\bar{N}_c+1}{2}} + b_1(\text{Tr } Y^2)^{\frac{\bar{N}_c-1}{2}} + \dots + b_{\frac{\bar{N}_c-1}{2}}(\text{Tr } Y^2) = 0 \quad (6.14)$$

There are two cases to discuss:

1. $N_c > N_f > \bar{N}_c$. In this case the electric theory is more strongly coupled than the magnetic one. The relation (6.14) is non trivial in the electric chiral ring, reducing the number of distinct vacua from $(N_c + 1)/2$ to $(\bar{N}_c + 1)/2$. This is in perfect agreement with the counting of vacua (6.5) which satisfy $r_1, r_2 \leq N_f$. The expectation values calculated from (6.14) agree with (6.5) for the appropriate vacua.
2. $\bar{N}_c > N_f > N_c$. The electric theory is more weakly coupled than the magnetic one. The relation (6.14) is satisfied on all electric vacua satisfying the classical relation (6.12) and therefore the electric moduli space is not modified quantum mechanically. This is in agreement with weak coupling intuition in the electric theory.

6.2. The case $\bar{N}_c = kN_f - N_c = 1$.

For $k = 1$ (SQCD with N_c colors and $N_f = N_c + 1$ flavors) the theory is known to confine [7]. The low energy degrees of freedom are the mesons M_j^i and baryons B_i, \tilde{B}^j . The classical constraint relating these fields is lifted quantum mechanically and is replaced by the superpotential:

$$W = \frac{1}{\Lambda^{2N_f-3}} \left(B_i M_j^i \tilde{B}^j - \det M \right). \quad (6.15)$$

This picture, which is due to strong coupling effects in the electric theory, is much more simply understood in the weakly coupled dual magnetic theory [6]. The mesons are the gauge singlets that appear in the dual. The baryons B_i, \tilde{B}^j are proportional to the magnetic quarks q_i, \tilde{q}^j . The first term in the superpotential (6.15) is the tree level magnetic superpotential (see e.g. (4.8)). The last term (proportional to $\det M$) is due to non-perturbative instanton effects that arise in the process of complete breaking of the magnetic gauge group.

For $k > 1$ we are in a position to use duality to make predictions. The operators generating the classical chiral ring of the $SU(kN_f - 1)$ gauge theory (2.3) are M_j (1.3), $\text{Tr } X^j, j = 2, \dots, k$, and the baryons (2.19),

$$\begin{aligned} B_i^{(1)} &= Q^{N_f-1} (XQ)^{N_f} \dots (X^{k-1}Q)^{N_f} \\ B_i^{(2)} &= Q^{N_f} (XQ)^{N_f-1} \dots (X^{k-1}Q)^{N_f} \\ &\dots\dots\dots \\ B_i^{(k)} &= Q^{N_f} (XQ)^{N_f} \dots (X^{k-1}Q)^{N_f-1} \end{aligned} \quad (6.16)$$

with $i = 1, \dots, N_f$, as well as antibaryons $\tilde{B}^{(j)}$. However, unlike the case $k = 1$, a long distance description which includes all these fields does not satisfy 't Hooft anomaly

matching. Duality suggests another solution to the problem. The only independent fields in a macroscopic description of the theory are M_j and $B^{(k)}, \tilde{B}^{(k)}$. The baryons $B^{(k)}, \tilde{B}^{(k)}$ which we will denote by B, \tilde{B} in this subsection, are mapped by duality (2.20) to the magnetic quarks. The other generators of the chiral ring either vanish or can be expressed in terms of M_j, B and \tilde{B} . For example, from (3.39) we deduce that:

$$\frac{1}{k+1-i} \text{Tr } X^{k+1-i} = \rho \tilde{B} M_{k-i} B + \frac{\partial \alpha_s}{\partial t_i}. \quad (6.17)$$

where $\rho = -(-)^{\frac{k(k-1)}{2} N_f} s_0^{-N_c-1} \Lambda^{1-(2k-1)N_c}$. Clearly the chiral ring at long distances is drastically modified from its classical structure. The superpotential of the theory can be read off (1.4):

$$W = s_0 \rho \tilde{B} M_k B \quad (6.18)$$

As a check note that the auxiliary scale μ present in (1.4) disappears in the electric variables M, B, \tilde{B} . The meson fields M_1, M_2, \dots, M_{k-1} do not appear in the superpotential. In general one expects additional dangerously irrelevant terms in the superpotential (6.18) which do depend on all M_j and whose effect would be to lift some flat directions and break some accidental symmetries. We have not analyzed these terms in detail.

The theory has rather different descriptions at short distances, where it is described in terms of gauge fields W_α , and quarks Q, X , and long distances where the quarks are confined and the appropriate degrees of freedom are the mesons M_j and baryons B, \tilde{B} . It is natural to ask what is the long distance, macroscopic description of deforming the superpotential W by the s_i , as in (2.7). The operator map (6.17) and deformed magnetic superpotential (3.35) with Y set to 0, tell us that in terms of the low energy degrees of freedom the structure is always essentially the same. The superpotential is

$$W = \rho \tilde{B} \left(\sum_{l=0}^{k-1} t_l M_{k-l} \right) B \quad (6.19)$$

The combination of mesons in brackets couples to the baryon field, and the other $k-1$ combinations of generalized mesons remain free.

This is in agreement with the microscopic picture, according to which turning on a generic superpotential (2.7) leads to the appearance of k distinct critical points a_i as described above. But because $N_c = \sum_i r_i = k N_f - 1$, there is a unique vacuum; we must assign $N_f - 1$ eigenvalues to one of the critical points (e.g. $r_1 = N_f - 1$) and N_f eigenvalues to each of the remaining $k-1$ ($r_i = N_f$ for $i \geq 2$). Thus, a microscopic physicist would

expect to see at low energies a set of almost decoupled SQCD theories, $k-1$ with $N_c = N_f$ and one with $N_c = N_f - 1$, coupled by irrelevant interactions. According to [7], in the present context (remembering the gauged $U(1)^{k-1}$ – see (2.10)), each of the theories with $N_f = N_c$ gives rise to a free meson field, whereas the factor with $N_f = N_c + 1$ gives rise to a meson and baryon coupled via the superpotential (6.15).

The combination of mesons in brackets in eq. (6.19) corresponds to the meson of the SQCD with $N_f = N_c + 1$. The baryons B, \tilde{B} correspond to the appropriate baryons of that theory. The other $k-1$ combinations of generalized mesons give the $k-1$ free mesons needed for the theories with $N_f = N_c$. Comparing to (6.15) we learn that (6.19) is missing a term proportional to $\det(\sum_l t_l M_{k-l})$ which is allowed by symmetries and can therefore appear much like in SQCD in the process of completely breaking the magnetic gauge group. Its coefficient in the superpotential is constrained by the symmetries to be a polynomial of the form $t_{k-1} + \dots + t_1^{k-1}$. We leave the detailed understanding of this term for future work.

It is not difficult to repeat the analysis above for the case when the microscopic theory is deformed by (2.7) with s_i fine tuned such that some of the critical points a_i coincide. In fact, by requiring the consistency of the resulting superpotential with all $\{s_i\}$ deformations puts extremely strong constraints on the corrections to the superpotential (6.19) and probably (over-) determines it. It would be interesting to find this superpotential.

6.3. The case $\bar{N}_c = kN_f - N_c = 2$.

The theory with $N_c = kN_f - 2$ is the simplest theory with a non-Abelian dual gauge group – $SU(2)$. Consider first the case $k = 2$. The electric theory has gauge group $G = SU(2N_f - 2)$ and superpotential:

$$W = \frac{s_0}{3} \text{Tr } X^3. \quad (6.20)$$

The magnetic one has gauge group $SU(2)$ and:

$$W_{\text{mag}} = \frac{s_0}{\mu^2} (M_1 \tilde{q} Y q + M_2 \tilde{q} q). \quad (6.21)$$

The magnetic theory has no superpotential for Y (setting other fields to their expectation values). Thus, there is a flat direction corresponding to

$$\langle Y \rangle = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.22)$$

Along it the gauge group is broken: $SU(2) \rightarrow U(1)$. Y is massive except for one massless field, $\text{Tr } Y^2$, parametrizing small fluctuations of the expectation value a .

Remarkably, the electric theory also has a flat direction, since N_c (which is even) is divisible by k ($= 2$). This flat direction, which was discussed in 6.1 is described by:

$$\langle X \rangle = \text{diag} (b^{N_f-1}, (-b)^{N_f-1}). \quad (6.23)$$

It too breaks the (electric) gauge symmetry:

$$SU(2N_f - 2) \rightarrow SU(N_f - 1) \times SU(N_f - 1) \times U(1). \quad (6.24)$$

X is massive except for one massless field, $\text{Tr } X^2$, parametrizing small fluctuations of b . The non-Abelian dynamics in (6.24) is SQCD with $N_f = N_c + 1$. We expect two sets of mesons and baryons coupled by the superpotential (6.15). Substituting (6.22) in (6.21) we indeed find the required structure. Linear combinations of the two meson fields M_1, M_2 couple to the two components of q, \tilde{q} which serve as baryons. The couplings look like the first term in (6.15). We are again not analyzing the $\det M$ terms due to instantons that must be added to the magnetic superpotential (6.21) in the weakly coupled magnetic theory. The $U(1)$ dynamics also agrees and the massless field $\text{Tr } X^2$ corresponds to $\text{Tr } Y^2$ completing the duality map.

At the origin ($a = b = 0$ in (6.22), (6.23)) we find a strongly coupled dual of the $SU(2)$ theory with $W = 0$.

For even $k > 2$ the structure is similar but richer. Theory space has many components depending on the number of distinct critical points. Consider, for example, the case when all k critical points are distinct. Since we want the magnetic superpotential for Y to vanish we add only odd powers of X to the electric superpotential (2.7) (i.e. all s_{2l+1} vanish). Therefore, the critical points appear in pairs, $\pm a_1, \pm a_2, \dots, \pm a_{\frac{k}{2}}$. There is a unique vacuum with

$$SU(kN_f - 2) \rightarrow [SU(N_f - 1)]^2 \times [SU(N_f)]^{k-2}. \quad (6.25)$$

The magnetic $SU(2)$ gauge theory is coupled to k singlet fields $M_j, j = 1, \dots, k$ and

$$W_{\text{mag}} = \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q.$$

There is still a flat direction (6.22), along which $k - 2$ combinations of M_j do not couple (in the IR at the origin of moduli space $\langle M_j \rangle = 0$), while two combine as before with

$q, \tilde{q} \rightarrow B, \tilde{B}$ to give the right structure for two SQCD theories with $N_f = N_c + 1$. The non-renormalizable part of the superpotential again remains to be analyzed. It is not hard to repeat these considerations for the case when some eigenvalues a_i coincide. We leave the details to the reader.

It is appropriate to end this section by returning to the truncation of the chiral ring, and explain why the quantum chiral ring is smaller than the classical one. In the example discussed here, in the quantum chiral ring there are relations:

$$\text{Tr } X^{2n+1} = \text{const}; \quad \text{Tr } X^{2n} = F(\text{Tr } X^2) \quad (6.26)$$

which are absent classically.

Consider first the classical theory where as usual we deform W (2.7) to resolve the singularity. One way of seeing that $\text{Tr } X^j$, $j = 2, \dots, k$ are all independent classically is to note that the theory with generic $\{s_i\}$ (2.7) has many classical vacua in which $\langle \text{Tr } X^j \rangle$ are all different. There are no relations among the $\langle \text{Tr } X^j \rangle$ that hold uniformly in all vacua. The key point is that as discussed previously many of the classical vacua are unstable and disappear in the quantum theory, due to the $N_f \geq N_c$ constraint in the low energy SQCD theories (see section 2). As we saw, in the case $N_c = kN_f - 2$ analyzed here most of the vacua disappear, leaving behind a unique vacuum (for generic s_i). Relations among the operators need to hold only in the quantum vacua, and therefore many more exist in general quantum mechanically than classically. One can easily convince oneself that the relations (6.26) in particular are valid in all the vacua that are stable quantum mechanically and break down in vacua that are unstable. Therefore they only exist in the quantum theory and not in the classical one.

7. Comments on the theory without a superpotential

The $SU(N_c)$ theory with an adjoint X , N_f fundamentals Q and N_f anti-fundamentals \tilde{Q} without a tree level superpotential is very interesting. Unfortunately, our understanding of this theory is very limited. In the previous sections we followed [8,9] and added the tree level superpotential $\frac{1}{k+1}s_0 X^{k+1}$ to simplify the analysis. In this section we will point out a few observations about the theory without the superpotential ($s_0 = 0$).

7.1. The moduli space

No dynamically generated superpotential can lift the classical flat directions. To see that, note that at a generic point in the moduli space the gauge group is broken to an Abelian subgroup (or completely broken). Therefore, instanton methods should be reliable. Repeating the analysis of [15], it is easy to see that instantons have too many fermion zero modes to generate a superpotential. Therefore, no superpotential can be generated and the theory has a moduli space of stable vacua for any N_f . The same conclusion can be reached by examining the symmetries and the way they restrict a dynamically generated superpotential.

Furthermore, the perturbations of this theory by various tree level superpotentials (see below) relate it to other theories where the moduli space is understood. This leads to further constraints on the quantum moduli space and its singularities. The conclusion is that the quantum moduli space is identical to the classical one. The only singularities are at points where classically the unbroken gauge symmetry is enhanced. The most singular point is at the origin. This does not mean that the physical interpretation of the singularities in the quantum theory is as in the classical theory.

7.2. The non-Abelian Coulomb phase $N_f \lesssim 2N_c$

When $N_f \lesssim 2N_c$ a simple two loop calculation similar to that in [17] reveals a non-trivial fixed point of the beta function. This calculation can be justified rigorously at large N_c . Therefore, the origin is in a non-Abelian Coulomb phase. The physics at that point is similar to the physics seen in perturbation theory, describing interacting quarks and gluons.

The dimensions of the chiral operators are constrained by the superconformal algebra to satisfy $D = \frac{3}{2}R$ where R is the charge of the $U(1)_R$ symmetry in the superconformal algebra. In simpler examples this $U(1)_R$ symmetry is easily identified [6]. In our case, it is ambiguous. Anomaly freedom constrains the R charge of Q and \tilde{Q} , B_f and the R charge of X , B_a to satisfy (2.2) $N_f B_f + N_c B_a = N_f$. For $\epsilon = \frac{2N_c - N_f}{2N_c} \ll 1$ the fixed point is visible in perturbation theory with the conclusion $B_a = \frac{2}{3}(1 - \epsilon \frac{N_c^2}{2N_c^2 - 1} + \mathcal{O}(\epsilon^2))$; $B_f = \frac{2}{3}(1 - \frac{\epsilon}{2} \frac{N_c^2 - 1}{2N_c^2 - 1} + \mathcal{O}(\epsilon^2))$.

As N_f is reduced, the fixed point becomes more strongly coupled and the dimensions of the operators become smaller. Since the dimensions of the spin zero fields cannot be smaller than one [20], at some point this description must be modified. We would expect,

by analogy with [6], that these fields become free and decouple. Then, the $U(1)_R$ in the superconformal algebra is an accidental symmetry which assigns $R = \frac{2}{3}$ to these fields. This observation might suggest that all $u_j = \text{Tr } X^j$ with $j = 2, \dots, N_c$ should be included as elementary fields in a dual description.

7.3. The confining phase superpotential

A useful tool in analyzing the dynamics of supersymmetric theories is the confining phase superpotential. It is obtained by coupling the generators of the chiral ring to external sources and computing the effective action for these sources. Upon a Legendre transform this leads to an effective action for these composites which gives a good description of the confining phase of the theory [21,2]. This procedure was used in [21] to study the $N_c = 2$, $N_f = 1$ theory. The authors of [22] studied the $N_c = 2$ problem for larger values of N_f and gave partial answers for larger N_c . The massless modes at the generic point in the moduli space are among these fields and therefore this effective superpotential gives a good description of the theory on the moduli space. The effective superpotential derived this way exhibits a singularity at the origin for every value of N_c and N_f . This means that at the origin more degrees of freedom are needed. This fact is in accord with our interpretation of the origin as being in a non-Abelian Coulomb phase. However, we could not find a useful description of the theory at the origin which gives rise to the singularity in these effective superpotentials.

7.4. The Coulomb phase

The theory with a tree level superpotential $\tilde{\lambda}_i^i \tilde{Q}_i X Q^i = \text{Tr } \lambda M_2$ can be analyzed easily. For $\tilde{\lambda}_i^i = \tilde{\delta}_i^i$ the theory becomes $N = 2$ supersymmetric. This theory for $N_c = 2$ was analyzed in [5] and for larger values of N_c in [23-25]. The moduli space of the theory has a Coulomb branch which has only massless photons at generic points. At special singular points on the moduli space there are more massless particles: massless monopoles, massless dyons, massless gluons and quarks, and even points with interacting non-trivial $N = 2$ superconformal field theories [26,27]. More quantitatively, this branch of the theory is described in terms of a hyperelliptic curve. The characteristic scale on this Coulomb branch is the only dimensionful parameter in the theory Λ which appears as a parameter in the curve.

It is easy to extend the curve away from this $N = 2$ theory; i.e. for arbitrary values of $\tilde{\lambda}_i^i \neq \tilde{\delta}_i^i$. As in [21,22,23] using the symmetries of the theory, this is achieved by replacing

every factor of $\Lambda^{2N_c-N_f}$ in the curve by $(\det \lambda)\Lambda^{2N_c-N_f}$. Therefore, as $\lambda \rightarrow 0$, all the features on the Coulomb branch approach the origin (more precisely, they approach points where classically there is an unbroken non-Abelian gauge symmetry). Conversely, by turning on λ the singularity at the origin splits to several singularities with various massless particles. Since these particles are not all local with respect to one another, there is no local Lagrangian which includes all of them. Therefore, it is impossible to write a local field theory which describes the deformation by λ along the entire Coulomb branch in weak coupling.

A similar situation was encountered in $SO(N_c)$ gauge theories [19,28]. There the theory at the origin was given several different dual descriptions. Each gave a weak coupling description of another deformation or another region of the moduli space. An attempt to imitate this procedure here will necessarily lead to a very large number of dual theories to describe the different phenomena in the Coulomb branch.

7.5. Deformation by a superpotential $\frac{s_0}{k+1}\text{Tr } X^{k+1}$

This is the theory we studied in this paper. Removing this perturbation by letting s_0 go to zero is a singular operation as it changes the asymptotic behavior of the potential. This can also be seen from the matching equation $\Lambda^{2N_c-N_f}\bar{\Lambda}^{2\bar{N}_c-N_f} = \left(\frac{\mu}{s_0}\right)^{2N_f}$ which becomes singular as s_0 goes to zero. Alternatively, we might attempt to remove this perturbation by letting k go to infinity.

As k becomes large, the operator $\text{Tr } X^{k+1}$ becomes irrelevant at the long distance theory at the origin and therefore it does not affect the dynamics. As we noted above, this operator is dangerously irrelevant and cannot be ignored. However, as k goes to infinity the potential it leads to becomes very flat and it is reasonable that the theory without a superpotential is achieved.

For large k the dual gauge group $SU(\bar{N}_c = kN_f - N_c)$ becomes large. This theory is strongly coupled and therefore one might think that it does not lead to a useful dual description. However, as we saw in the previous sections, this theory gives a weak coupling description of some of the deformations. Therefore, if there is a unique good dual theory at the origin, it should include this $SU(\bar{N}_c)$ for arbitrarily large \bar{N}_c . Such an $SU(\infty)$ theory is expected to behave like a string theory. Therefore, we might speculate that the dual theory at the origin is not a field theory but a string theory. Perhaps, if this is indeed the case, there will not be a need for a large number of dual descriptions as suggested by the structure of the Coulomb phase.

Acknowledgments

We would like to thank T. Banks, K. Intriligator, S. Shenker, and especially E. Witten for many helpful discussions. This work was supported in part by DOE grant #DE-FG05-90ER40559, BSF grant number 5360/2, the Minerva foundation and a DOE OJI grant. A.S. would like to thank the hospitality of the Enrico Fermi Institute where part of this work was performed. D.K. thanks the Weizmann Institute, Aspen Physics Center and Department of Physics at Rutgers University for hospitality during the course of this work.

Appendix A. Dangerously irrelevant operators

A quantum field theorist might wonder about the presence of high order polynomials such as (2.3) in the superpotential. These are non-renormalizable interactions which seem irrelevant for the long distance behavior of the theory. How is it then that they affect the physical results? In order to answer this question we should review the notion of a dangerously irrelevant (DI) operator in the theory of the renormalization group.

Consider deformations of a fixed point of the renormalization group by relevant operators or along flat directions of the potential. It might be that an irrelevant operator at the original fixed point becomes relevant after the deformation. Such an operator is called dangerously irrelevant; ignoring its presence will lead to incorrect results.

An example of a dangerously irrelevant operator is the gauge superfield $W_\alpha W^\alpha$ in a non asymptotically free gauge theory (for definiteness one may think of supersymmetric QCD with quarks in the fundamental representation of the gauge group). In such a situation the gauge coupling flows to zero at long distance; hence the operator $W_\alpha W^\alpha$ is irrelevant. Nevertheless, it is clearly important to keep the gauge coupling when describing the long distance behavior of gauge theories. If, for example, we turn on quark masses, the number of light quarks may fall below the asymptotic freedom bound and therefore the gauge coupling becomes relevant. Thus, W_α^2 is relevant in part of theory space and irrelevant in other parts. When it is irrelevant it is referred to as being dangerously irrelevant.

The operator $\text{Tr } X^{k+1}$ behaves in a very similar way. At the gaussian (UV) fixed point it has (for $k > 2$) dimension larger than three. However, if we turn on the gauge coupling g , $\text{Tr } X^{k+1}$ develops for sufficiently small N_f an anomalous dimension which can make it relevant. Hence, while this operator is irrelevant near the gaussian UV fixed point, it too is a dangerously irrelevant operator.

Actually, $\text{Tr } X^{k+1}$ is dangerously irrelevant even without the gauge coupling. While it is irrelevant when expanding around the trivial, $X = 0$ vacuum, it is clearly relevant when expanding around any non-zero X . Thus, the presence of the superpotential (2.3) lifts some of the flat directions of the original theory; also, turning on a polynomial superpotential (1.1) one finds many minima at non vanishing X , where clearly all powers up to and including $\text{Tr } X^{k+1}$ are relevant. Therefore, at the origin $\text{Tr } X^{k+1}$ is a DI operator and cannot be ignored.

Appendix B. Properties of symmetric polynomials.

Consider a polynomial of order $k + 1$,

$$W(x) = \sum_{i=0}^k \frac{1}{k+1-i} s_i x^{k+1-i} \quad (\text{B.1})$$

The k roots of $W'(x)$, a_i ($i = 1, \dots, k$) satisfy:

$$W' = \sum_{i=0}^k s_i x^{k-i} \equiv s_0 \prod_{i=1}^k (x - a_i) \quad (\text{B.2})$$

where $\{s_l\}$ and $\{a_i\}$ are related by:

$$s_l = (-)^l s_0 \sum_{i_1 < i_2 < \dots < i_l} a_{i_1} a_{i_2} \dots a_{i_l} \quad (\text{B.3})$$

It is also natural to define the objects:

$$u_l \equiv \sum_{i=1}^k a_i^l \quad (\text{B.4})$$

which satisfy the recursion relation:

$$l s_l + \sum_{i=1}^l s_{l-i} u_i = 0; \quad l = 1, 2, 3, \dots \quad (\text{B.5})$$

Eq. (B.5) can be thought of as determining u_l in terms of $\{s_m\}$ with $m \leq l$. For the few lowest cases we have:

$$\begin{aligned} u_1 &= -\frac{s_1}{s_0} \\ u_2 &= \left(\frac{s_1}{s_0}\right)^2 - 2\left(\frac{s_2}{s_0}\right) \\ u_3 &= 3\frac{s_2}{s_0}\frac{s_1}{s_0} - \left(\frac{s_1}{s_0}\right)^3 - 3\frac{s_3}{s_0} \end{aligned} \quad (\text{B.6})$$

etc. In the text (section 3) we used a remarkable property of the u 's:

$$\frac{\partial}{\partial s_j} \frac{u_{j+l}}{j+l} = \frac{\partial}{\partial s_i} \frac{u_{i+l}}{i+l} \quad (\text{B.7})$$

For all j, l . In other words, duality requires that $\frac{\partial}{\partial s_j} \frac{u_{j+l}}{j+l}$ should be independent of j . The proof of this statement relies on the following representation of the recursion relation (B.5):

$$\sum_n \frac{t^n}{n} u_n = \ln \left(1 + \sum_i s_i t^i \right) \quad (\text{B.8})$$

Differentiating (B.8) w.r.t. some s_j we find:

$$\sum_n \frac{t^{n-j}}{n} \frac{\partial}{\partial s_j} u_n = \frac{1}{1 + \sum_i s_i t^i} \quad (\text{B.9})$$

which makes the fact that $\frac{\partial}{\partial s_j} u_n / n$ depends only on $n - j$ and not on n, j separately manifest.

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